

On Formality and Solvmanifolds



I n a u g u r a l - D i s s e r t a t i o n

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

Christoph Bock

Köln

2009

Berichterstatter: Prof. Hansjörg Geiges, Ph.D. (Cantab)
Prof. Dr. George Marinescu

Tag der letzten mündlichen Prüfung: 4. Februar 2009

Die vorliegende Version dieser Dissertation enthält einige Korrekturen, die im Sommer 2009 eingefügt wurden.

The present version of this thesis contains some corrections that have been included in Summer 2009.

Für Nina

Vorwort

Bis in die siebziger Jahre war nicht bekannt, ob es kompakte symplektische Mannigfaltigkeiten gibt, die keine Kähler-Struktur tragen. Das erste Beispiel einer solchen Mannigfaltigkeit wurde 1976 von W. P. Thurston angegeben. Er konstruierte in [73] eine symplektische vier-dimensionale Nilmannigfaltigkeit (d.i. ein kompakter Quotient einer zusammenhängenden und einfach-zusammenhängenden nilpotenten Liegruppe nach einer diskreten Untergruppe) mit erster Betti-Zahl $b_1 = 3$. Aus topologischen Gründen kann diese Mannigfaltigkeit nicht Kählersch sein, denn die Betti-Zahlen b_{2i+1} von ungeradem Grad sind für Kähler-Mannigfaltigkeiten gerade. L. A. Cordero, M. Fernández und A. Gray haben in den achtziger Jahren weitere Beispiele angegeben (vgl. [13]), die aber teilweise gerade Betti-Zahlen haben. Die Autoren weisen nach, daß ihre Beispiele nicht formal sind. Hieraus folgt dann, daß sie auch nicht Kählersch sein können, denn P. Deligne, P. Griffiths, J. Morgan und D. Sullivan haben in [16] bewiesen, daß Formalität notwendig für die Existenz von Kähler-Strukturen ist.

Formalität ist eine wichtige Eigenschaft eines Raumes, die es ermöglicht, rational-homotopische Informationen aus der Kohomologie-Algebra zu gewinnen. Die o.g. Arbeit [13] zeigt insbesondere, daß symplektische Mannigfaltigkeiten i.a. nicht formal sind. Außerdem stellen Methoden der rationalen Homotopie Möglichkeiten bereit, kompakte symplektische nicht-Kählersche Mannigfaltigkeiten zu konstruieren.

Es sei angemerkt, daß es formale symplektische Mannigfaltigkeiten gibt, die den Kohomologietyp einer Kähler-Mannigfaltigkeit haben und trotzdem nicht Kählersch sind. Ein Beispiel hierfür haben M. Fernández und A. Gray [25] gegeben.

Ich gebe im ersten Kapitel dieser Arbeit einen kurzen Überblick über die Theorie der minimalen Modelle, insoweit sie zur Definition des Begriffes der Formalität notwendig ist.

M. Fernández und V. Muñoz haben in dieser Dekade eine Arbeit [28] über die Geographie formaler Mannigfaltigkeiten geschrieben. In Abhängigkeit der Dimension und der ersten Betti-Zahl b_1 sagen sie genau, wann eine geschlossene formale Mannigfaltigkeit existiert. Im zweiten Kapitel versuche ich, dieselbe Fragestellung für Mannigfaltigkeiten, die zusätzlich eine symplektische Struktur tragen, zu klären. Dies ist zunächst nur mit Ausnahme des sechs-dimensionalen Falles

mit $b_1 = 1$ gelungen. Ferner kann man auch eine Aussage über die Geographie formaler geschlossener Kontaktmannigfaltigkeiten mit erster Betti-Zahl größer oder gleich zwei herleiten. Dies stelle ich im zweiten Kapitel ebenfalls dar.

In der Hoffnung, den offen gebliebenen Fall des sechs-dimensionalen Raumes mit $b_1 = 1$ beantworten zu können, habe ich mich dem Studium von Solvmannigfaltigkeiten zugewendet, welches den Inhalt des dritten Kapitels bildet.

Nilmannigfaltigkeiten stellen eine reichhaltige Quelle symplektischer Mannigfaltigkeiten, die nicht Kählersch sind, dar. Tatsächlich ist eine Nilmannigfaltigkeit genau dann formal, wenn sie ein Torus ist. (Und genau in diesem Fall trägt sie auch eine Kähler-Struktur.) M. a. W. ist jede symplektische nicht-torale Nilmannigfaltigkeit nicht formal. Nilmannigfaltigkeiten helfen bei der Suche nach einer Mannigfaltigkeit mit $b_1 = 1$ jedoch nicht weiter, da b_1 für sie immer größer als eins ist. Der Begriff der Solvmannigfaltigkeit ist eine Verallgemeinerung desjenigen der Nilmannigfaltigkeit. Eine Solvmannigfaltigkeit ist ein kompakter Quotient aus einer zusammenhängenden und einfach-zusammenhängenden auflösbaren Liegruppe nach einer diskreten Untergruppe, und solche können auch erste Betti-Zahl gleich eins haben. Es erschien mir daher natürlich, unter den Solvmannigfaltigkeiten nach einem Beispiel einer nicht-formalen symplektischen sechs-Mannigfaltigkeit mit $b_1 = 1$ zu suchen.

In diesem Zusammenhang habe ich dann auch versucht, die bisher bekannte Klassifikation niedrig-dimensionaler Solvmannigfaltigkeiten bis zur Dimension sechs zu erweitern und den Aspekt der Formalität hinzuzufügen.

Im sechs-dimensionalen Fall habe ich mich auf die Betrachtung von symplektischen Räumen beschränkt und unter diesen eine nicht-formale Mannigfaltigkeit mit $b_1 = 1$ gefunden.

Neben der Formalität und der Tatsache, daß die Betti-Zahlen ungeraden Grades gerade sind, erfüllen kompakte Kähler-Mannigfaltigkeiten die sog. starke Lefschetz-Bedingung. Bezeichnet ω eine symplektische Form auf einer $2n$ -dimensionalen kompakten Mannigfaltigkeit M , so lautet die starke Lefschetz-Bedingung, daß das Cup-Produkt mit $[\omega]^k$ für alle $k \in \{0, \dots, n-1\}$ einen Isomorphismus $H^{n-k}(M, \mathbb{R}) \rightarrow H^{n+k}(M, \mathbb{R})$ definiert.

Zum Abschluß von Kapitel 3 gehe ich der Frage nach, welche Kombinationen der drei genannten Eigenschaften für symplektische Solvmannigfaltigkeiten erfüllt bzw. nicht erfüllt sein können und beantworte zwei Fragen, die in der Arbeit [47] von R. Ibáñez, Y. Rudiak, A. Tralle und L. Ugarte offen geblieben waren.

St. Halperin nennt in [37] ein Ergebnis, das die Berechnung der höheren Homotopiegruppen einer gewissen Klasse von Räumen, die die der nilpotenten umschließt, mittels der Theorie der minimalen Modelle ermöglicht. Den Nachweis dessen, der in [37] nicht dargestellt ist, werde ich im vierten Kapitel dieser Arbeit erbringen.

Sehr herzlich danke ich Herrn Prof. H. Geiges für die Möglichkeit, diese Arbeit unter seiner Anleitung zu schreiben. Ich habe von vielen anregenden Gesprächen

und wertvollen Hinweisen, die mir halfen, neue Aspekte zu berücksichtigen, profitiert. Seine Unterstützung, die Möglichkeit, ihm jederzeit Fragen zu stellen, und der gewährte Freiraum bei der Erstellung dieser Arbeit haben einen maßgeblichen Anteil an ihr.

Mein weiterer Dank gilt meinem Diplomvater Herrn Prof. Dr. W. Henke, der mich die Mathematik gelehrt und die Begeisterung für sie in mir geweckt hat.

Köln, im Dezember 2008

Christoph Bock

Abstract

Topology of symplectic manifolds is nowadays a subject of intensive development. The simplest examples of such manifolds are Kähler manifolds and an important property of the latter is their formality. Thus, a possible way of constructing symplectic manifolds with no Kähler structure is to find such ones which are not formal.

M. Fernández und V. Muñoz considered in [28] the question of the geography of non-formal compact manifolds. Given $(m, b_1) \in \mathbb{N}_+ \times \mathbb{N}$, they showed whether or not there are m -dimensional non-formal compact manifolds with first Betti number b_1 .

The aim of this thesis is to answer the same question for compact symplectic manifolds. After setting the scene in the first chapter, this is done in the second one – except for the six-dimensional case with $b_1 = 1$. The third chapter deals with solvmanifolds, especially with those of dimension less or equal to six, because I hoped to find the missing example among them, and in fact there is a six-dimensional symplectic solvmanifolds which is non-formal and satisfies $b_1 = 1$.

Besides formality, compact Kähler manifolds have even odd-degree Betti numbers and they satisfy the so-called Hard Lefschetz condition. To end Chapter 3, I deal with relations between this three properties for symplectic solvmanifolds. I am able to give an answer to two questions that had remained open in the article [47] of R. Ibáñez, Y. Rudiak, A. Tralle and L. Ugarte.

Furthermore, in the last chapter I prove a result that allows an easy computation of the higher homotopy groups of a class of spaces containing all nilpotent ones. Without giving a proof, St. Halperin stated it in the introduction of [37].

I would like to express my sincere gratitude to my supervisor Prof. H. Geiges for giving me the possibility to participate in his group and to write this thesis under his guidance. I have profited greatly from his suggestions and various conversations with him. Without his kind support this dissertation would not have been written.

Moreover, I wish to thank Prof. Dr. W. Henke who was the supervisor of my diploma thesis. He taught me mathematics and woke up the enthusiasm for it.

Köln, December 2008

Christoph Bock

Contents

1	Introduction	1
1.1	Differential graded algebras and formality	1
1.2	Massey products	5
1.3	Geography of non-formal manifolds	6
2	Symplectic and Contact Manifolds	9
2.1	Symplectic, Kähler and Lefschetz manifolds	9
2.2	Donaldson submanifolds	12
2.3	Known examples	14
2.3.1	The manifolds $M(p,q)$	14
2.3.2	The manifold $M_{8,0}$	15
2.4	Proofs	16
2.5	Contact manifolds	17
3	Solvmanifolds	21
3.1	Nilmanifolds	22
3.2	Solvmanifolds in general	25
3.3	Semidirect products	30
3.4	Semisimple splittings	35
3.5	Three-dimensional solvmanifolds	38
3.6	Four-dimensional solvmanifolds	43
3.7	Five-dimensional solvmanifolds	50
3.7.1	Nilpotent and decomposable solvable Lie algebras	50
3.7.2	Indecomposable non-nilpotent Lie algebras	51
3.7.3	Contact structures	64
3.8	Six-dimensional solvmanifolds	65
3.8.1	Nilmanifolds	66
3.8.2	Candidates for the existence of lattices	66
3.8.3	Symplectic solvmanifolds with $b_1 = 1$	68
3.8.4	Symplectic solvmanifolds with $b_1 > 1$	79
3.8.5	Decomposable solvmanifolds	88
3.9	Relations with the Lefschetz property	88

4 Rational Homotopy and Minimal Models	97
4.1 PL forms	97
4.2 Nilpotent spaces	98
4.3 The Halperin-Grivel-Thomas theorem	99
A Lists of Lie Algebras	103
B Integer Polynomials	119
C Group Extensions	125
Bibliography	129
Index	135

Chapter 1

Introduction

The aim of this chapter is to introduce formal manifolds and to quote some of their properties. (Throughout this thesis a manifold is assumed to have no boundary, i.e. a compact manifold is the same as a closed manifold.) Given a manifold M , one can consider the complex of its differential forms $(\Omega(M), d)$, which has the structure of a so-called differential graded algebra. Such differential graded algebras are the main objects of rational homotopy theory, and the definition of the formality of M will purely depend on rational homotopic properties of $(\Omega(M), d)$.

The idea of rational homotopy is to ignore the torsion in standard homotopy theory. Sullivan [71] showed in the 1960s that not only the simplicial homology $H_*(X, \mathbb{Z})$ and the higher homotopy groups $\pi_i(X)$, $i > 1$, of a simply-connected space X can be localised to $H_*(X, \mathbb{Q})$ and $\pi_i(X) \otimes \mathbb{Q}$. It is also possible to *geometrically* localise the space X to a space X_0 via a continuous map $X \rightarrow X_0$ which induces isomorphisms $H_*(X, \mathbb{Q}) \rightarrow H_*(X_0, \mathbb{Z})$ and $\pi_i(X) \otimes \mathbb{Q} \rightarrow \pi_i(X_0)$. The *rational homotopy type* of X is then defined as the weak homotopy type of X_0 . A principal feature of rational homotopy theory, as developed by Quillen [65], is that the *geometric* localisation X_0 can be understood within an entirely *algebraic* category. This led to Sullivan's choice [72] of a particular algebraic category that models exactly the rational homotopy type of a space. It is to this category – the category of minimal differential graded algebras – that we turn now.

1.1 Differential graded algebras and formality

Let \mathbb{K} be a field of characteristic zero. A *differential graded algebra (DGA)* is a graded \mathbb{K} -algebra $A = \bigoplus_{i \in \mathbb{N}} A^i$ together with a \mathbb{K} -linear map $d: A \rightarrow A$ such that $d(A^i) \subset A^{i+1}$ and the following conditions are satisfied:

- (i) The \mathbb{K} -algebra structure of A is given by an inclusion $\mathbb{K} \hookrightarrow A^0$.

- (ii) The multiplication is graded commutative, i.e. for $a \in A^i$ and $b \in A^j$ one has $a \cdot b = (-1)^{i \cdot j} b \cdot a \in A^{i+j}$.
- (iii) The Leibniz rule holds: $\forall_{a \in A^i} \forall_{b \in A} d(a \cdot b) = d(a) \cdot b + (-1)^i a \cdot d(b)$
- (iv) The map d is a differential, i.e. $d^2 = 0$.

Further, we define $|a| := i$ for $a \in A^i$.

The i -th cohomology of a DGA (A, d) is the algebra

$$H^i(A, d) := \frac{\ker(d: A^i \rightarrow A^{i+1})}{\text{im}(d: A^{i-1} \rightarrow A^i)}.$$

If (B, d_B) is another DGA, then a \mathbb{K} -linear map $f: A \rightarrow B$ is called *morphism* if $f(A^i) \subset B^i$, f is multiplicative, and $d_B \circ f = f \circ d_A$. Obviously, any such f induces a homomorphism $f^*: H^*(A, d_A) \rightarrow H^*(B, d_B)$. A morphism of differential graded algebras inducing an isomorphism on cohomology is called *quasi-isomorphism*.

Definition 1.1.1. A DGA (\mathcal{M}, d) is said to be *minimal* if

- (i) there is a graded vector space $V = \left(\bigoplus_{i \in \mathbb{N}_+} V^i\right) = \text{Span}\{a_k \mid k \in I\}$ with homogeneous elements a_k , which we call the generators,
- (ii) $\mathcal{M} = \bigwedge V$,
- (iii) the index set I is well ordered, such that $k < l \Rightarrow |a_k| \leq |a_l|$ and the expression for da_k contains only generators a_l with $l < k$.

We shall say that (\mathcal{M}, d) is a *minimal model for a differential graded algebra* (A, d_A) if (\mathcal{M}, d) is minimal and there is a quasi-isomorphism of DGAs $\rho: (\mathcal{M}, d) \rightarrow (A, d_A)$, i.e. ρ induces an isomorphism $\rho^*: H^*(\mathcal{M}, d) \rightarrow H^*(A, d_A)$ on cohomology.

The importance of minimal models is reflected by the following theorem, which is taken from Sullivan's work [72, Section 5].

Theorem 1.1.2. *A differential graded algebra (A, d_A) with $H^0(A, d_A) = \mathbb{K}$ possesses a minimal model. It is unique up to isomorphism of differential graded algebras.*

We quote the existence-part of Sullivan's proof, which gives an explicit construction of the minimal model. Whenever we are going to construct such a model for a given algebra in this thesis, we will do it as we do it in this proof.

Proof of the existence. We need the following algebraic operations to "add" resp. "kill" cohomology.

Let (\mathcal{M}, d) be a DGA. We “add” cohomology by choosing a new generator x and setting

$$\widetilde{\mathcal{M}} := \mathcal{M} \otimes \bigwedge(x), \quad \tilde{d}|_{\mathcal{M}} = d, \quad \tilde{d}(x) = 0,$$

and “kill” a cohomology class $[z] \in H^k(\mathcal{M}, d)$ by choosing a new generator y of degree $k - 1$ and setting

$$\widetilde{\mathcal{M}} := \mathcal{M} \otimes \bigwedge(y), \quad \tilde{d}|_{\mathcal{M}} = d, \quad \tilde{d}(y) = z.$$

Note that z is a polynomial in the generators of \mathcal{M} .

Now, let (A, d_A) a DGA with $H^0(A, d_A) = \mathbb{K}$. We set $\mathcal{M}_0 := \mathbb{K}$, $d_0 := 0$ and $\rho_0(x) = x$.

Suppose now $\rho_k: (\mathcal{M}_k, d_k) \rightarrow (A, d_A)$ has been constructed so that ρ_k induces isomorphisms on cohomology in degrees $\leq k$ and a monomorphism in degree $(k + 1)$.

“Add” cohomology in degree $(k + 1)$ to get a morphism of differential graded algebras $\rho_{(k+1),0}: (\mathcal{M}_{(k+1),0}, d_{(k+1),0}) \rightarrow (A, d_A)$ which induces an isomorphism $\rho_{(k+1),0}^*$ on cohomology in degrees $\leq (k + 1)$. Now, we want to make the induced map $\rho_{(k+1),0}^*$ injective on cohomology in degree $(k + 2)$.

We “kill” the kernel on cohomology in degree $(k + 2)$ (by non-closed generators of degree $(k + 1)$) and define $\rho_{(k+1),1}: (\mathcal{M}_{(k+1),1}, d_{(k+1),1}) \rightarrow (A, d_A)$ accordingly. If there are generators of degree one in $(\mathcal{M}_{(k+1),0}, d_{(k+1),0})$ it is possible that this killing process generates new kernel on cohomology in degree $(k + 2)$. Therefore, we may have to “kill” the kernel in degree $(k + 2)$ repeatedly.

We end up with a morphism $\rho_{(k+1),\infty}: (\mathcal{M}_{(k+1),\infty}, d_{(k+1),\infty}) \rightarrow (A, d_A)$ which induces isomorphisms on cohomology in degrees $\leq (k + 1)$ and a monomorphism in degree $(k + 2)$. Set $\rho_{k+1} := \rho_{(k+1),\infty}$ and $(\mathcal{M}_{k+1}, d_{k+1}) := (\mathcal{M}_{(k+1),\infty}, d_{(k+1),\infty})$.

Inductively we get the minimal model $\rho: (\mathcal{M}, d) \rightarrow (A, d_A)$. \square

A *minimal model* (\mathcal{M}_M, d) of a *connected smooth manifold* M is a minimal model for the de Rahm complex $(\Omega(M), d)$ of differential forms on M . Note that this implies that (\mathcal{M}_M, d) is an algebra over \mathbb{R} . The last theorem implies that every connected smooth manifold possesses a minimal model which is unique up to isomorphism of differential graded algebras.

For a certain class of spaces that includes all nilpotent (and hence all simply-connected) spaces, we can read off the non-torsion part of the homotopy from the generators of the minimal model. We point the interested reader to Chapter 4.

Except in Chapter 4, we are from now on just considering differential graded algebras over the field $\mathbb{K} = \mathbb{R}$.

For the remainder of this section, we deal with the notion of formality. Endowed with the trivial differential, the cohomology of a minimal DGA is a DGA itself, and therefore it also possesses a minimal model. In general, these two minimal models need not to be isomorphic.

A minimal differential graded algebra (\mathcal{M}, d) is called *formal* if there is a morphism of differential graded algebras

$$\psi: (\mathcal{M}, d) \longrightarrow (H^*(\mathcal{M}, d), d_H = 0)$$

that induces the identity on cohomology.

This means that (\mathcal{M}, d) and $(H^*(\mathcal{M}, d), d_H = 0)$ share their minimal model. The following theorem gives an equivalent characterisation.

Theorem 1.1.3 ([63, Theorem 1.3.1]). *A minimal model (\mathcal{M}, d) is formal if and only if we can write $\mathcal{M} = \bigwedge V$ and the space V decomposes as a direct sum $V = C \oplus N$ with $d(C) = 0$, d is injective on N , and such that every closed element in the ideal $I(N)$ generated by N in $\bigwedge V$ is exact.* \square

This allows us to give a weaker version of the notion of formality.

Definition 1.1.4. A minimal model (\mathcal{M}, d) is called *s-formal*, $s \in \mathbb{N}$, if we can write $\mathcal{M} = \bigwedge V$ and for each $i \leq s$ the space V^i generated by generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$ with $d(C^i) = 0$, d is injective on N^i and such that every closed element in the ideal $I(\bigoplus_{i \leq s} N^i)$ generated by $\bigoplus_{i \leq s} N^i$ in $\bigwedge (\bigoplus_{i \leq s} V^i)$ is exact in $\bigwedge V$.

Obviously, formality implies s -formality for every s .

The following theorem is an immediate consequence of the last definition.

Theorem 1.1.5. *Let (\mathcal{M}, d) be a minimal model, where $\mathcal{M} = \bigwedge V$, $V = C \oplus N$ with $d(C) = 0$ and d is injective on N .*

Assume that there exist $r, s \in \mathbb{N}_+$, $n \in \mathbb{N}^r$ and $x \in \bigwedge (\bigoplus_{i \leq s} V^i)$ such that holds

$$\forall_{c \in C^r} (n + c)x \text{ is closed and not exact.}$$

Then (\mathcal{M}, d) is not $\max\{r, s\}$ -formal. \square

A connected smooth manifold is called *formal* (resp. *s-formal*) if its minimal model is formal (resp. s -formal).

Example ([63, p. 20]). Any compact Riemannian symmetric space is formal. \square

We end this section with some results that allow an easier detection of formality resp. non-formality. The next theorem shows the reason for defining s -formality: in certain cases s -formality is sufficient for a manifold to be formal.

Theorem 1.1.6 ([27, Theorem 3.1]). *Let M be a connected and orientable compact smooth manifold of dimension $2n$ or $(2n - 1)$.*

Then M is formal if and only if it is $(n - 1)$ -formal. \square

Example ([27, Corollary 3.3]).

- (i) Every connected and simply-connected compact smooth manifold is 2-formal.
- (ii) Every connected and simply-connected compact smooth manifold of dimension seven or eight is formal if and only if it is 3-formal. \square

Proposition 1.1.7 ([27, Lemma 2.11]). *Let M_1, M_2 be connected smooth manifolds. They are both formal (resp. s -formal) if and only if $M_1 \times M_2$ is formal (resp. s -formal).* \square

1.2 Massey products

An important tool for detecting non-formality is the concept of Massey products: As we shall see below, the triviality of the Massey products is necessary for formality.

Let (A, d) be a differential graded algebra.

- (i) Let $a_i \in H^{p_i}(A, d)$, $p_i > 0$, $1 \leq i \leq 3$, satisfying $a_j \cdot a_{j+1} = 0$ for $j = 1, 2$. Take elements α_i of A with $a_i = [\alpha_i]$ and write $\alpha_j \cdot \alpha_{j+1} = d\xi_{j,j+1}$ for $j = 1, 2$. The (*triple-*)*Massey product* $\langle a_1, a_2, a_3 \rangle$ of the classes a_i is defined as

$$[\alpha_1 \cdot \xi_{2,3} + (-1)^{p_1+1} \xi_{1,2} \cdot \alpha_3] \in \frac{H^{p_1+p_2+p_3-1}(A, d)}{a_1 \cdot H^{p_2+p_3-1}(A, d) + H^{p_1+p_2-1}(A, d) \cdot a_3}.$$

- (ii) Now, let $k \geq 4$ and $a_i \in H^{p_i}(A, d)$, $p_i > 0$, $1 \leq i \leq k$, such that $\langle a_1, \dots, a_{k-1} \rangle$ and $\langle a_2, \dots, a_k \rangle$ are defined and vanish simultaneously, i.e. there are elements $\xi_{i,j}$ of A , $1 \leq i \leq j \leq k$, $(i, j) \neq (1, k)$, such that

$$a_i = [\xi_{i,i}] \quad \text{and} \quad d\xi_{i,j} = \sum_{l=i}^{j-1} \bar{\xi}_{i,l} \cdot \xi_{l+1,j}, \quad (1.1)$$

where $\bar{\xi} = (-1)^{|\xi|} \xi$. The *Massey product* $\langle a_1, \dots, a_k \rangle$ of the classes a_i is defined as the set $\{[\sum_{l=1}^{k-1} \bar{\xi}_{1,l} \cdot \xi_{l+1,k}] \mid \xi_{i,j} \text{ satisfies (1.1)}\}$. This is a subset of $H^{p_1+\dots+p_k-(k-2)}(A, d)$.

We say that $\langle a_1, \dots, a_k \rangle$ vanishes if $0 \in \langle a_1, \dots, a_k \rangle$.

Remark. The definition of the triple-Massey product in (i) as an element of a quotient space is well defined, see e.g. [63, Section 1.6].

The next two lemmata show the relation between formality (resp. s -formality) and Massey products.

Lemma 1.2.1 ([63, Theorem 1.6.5]). *For any formal minimal differential graded algebra all Massey products vanish.* \square

Lemma 1.2.2 ([27, Lemma 2.9]). *Let (A, d) be an s -formal minimal differential graded algebra. Suppose that there are cohomology classes $a_i \in H^{p_i}(A, d)$, $p_i > 0$, $1 \leq i \leq k$, such that $\langle a_1, \dots, a_k \rangle$ is defined. If $p_1 + \dots + p_{k-1} \leq s + k - 2$ and $p_2 + \dots + p_k \leq s + k - 2$, then $\langle a_1, \dots, a_k \rangle$ vanishes.* \square

In [29], Fernández and Muñoz introduce a different type of Massey product, called G-Massey product:

Definition 1.2.3. Let (A, d) be a DGA and let $a, b_1, b_2, b_3 \in H^2(A, d)$ satisfying $a \cdot b_i = 0$ for $i = 1, 2, 3$. Take choices of representatives $a = [a]$, $b_i = [\beta_i]$ and $\alpha \cdot \beta_i = d\xi_i$ for $i = 1, 2, 3$. Then the *G-Massey product* $\langle a; b_1, b_2, b_3 \rangle$ is defined as $[\xi_1 \cdot \xi_2 \cdot \beta_3 + \xi_2 \cdot \xi_3 \cdot \beta_1 + \xi_3 \cdot \xi_1 \cdot \beta_2]$ in

$$\frac{H^8(A, d)}{\langle b_1, a, b_2 \rangle \cdot H^3(A, d) + \langle b_1, a, b_3 \rangle \cdot H^3(A, d) + \langle b_2, a, b_3 \rangle \cdot H^3(A, d)}.$$

Lemma 1.2.4 ([29, Proposition 3.2]). *If a minimal differential graded algebra is formal, then every G-Massey product vanishes.* \square

Corollary 1.2.5. *If the de Rham complex $(\Omega(M), d)$ of a smooth manifold M possesses a non-vanishing Massey or G-Massey product, then M is not formal.*

If there are cohomology classes $a_i \in H^{p_i}(M, \mathbb{R})$ ($p_i > 0$, $1 \leq i \leq k$) with $p_1 + \dots + p_{k-1} \leq s + k - 2$ and $p_2 + \dots + p_k \leq s + k - 2$ such that $\langle a_1, \dots, a_k \rangle$ does not vanish, then M is not s -formal.

Proof. This holds since a minimal model $\rho: (\mathcal{M}_M, d) \rightarrow (\Omega(M), d)$ induces an isomorphism on cohomology. \square

1.3 Geography of non-formal manifolds

Fernández and Muñoz considered in [28] the geography of non-formal compact manifolds. This means they examined whether there are non-formal compact manifolds of a given dimension with a given first Betti number. They obtained the following theorem:

Theorem 1.3.1. *Given $m \in \mathbb{N}_+$ and $b \in \mathbb{N}$, there are compact oriented m -dimensional smooth manifolds with $b_1 = b$ which are non-formal if and only if one of the following conditions holds:*

(i) $m \geq 3$ and $b \geq 2$,

(ii) $m \geq 5$ and $b = 1$,

(iii) $m \geq 7$ and $b = 0$. \square

A natural question to ask is when there are non-formal compact symplectic or contact manifolds as in the last theorem. Parts of this question will be answered in the next chapter.

Chapter 2

Geography of Non-Formal Symplectic and Contact Manifolds

In this chapter we want to construct non-formal symplectic and contact m -manifolds. If b denotes the first Betti number, then the pair (m, b) must satisfy one of the conditions (i), (ii) or (iii) of Theorem 1.3.1. Unfortunately, we shall not find examples for all possible pairs (m, b) . But we will be able to prove that the geography of even-dimensional compact manifolds coincides with that of compact symplectic manifolds.

2.1 Symplectic, Kähler and Lefschetz manifolds

The main examples of formal spaces are Kähler manifolds. By definition, a Kähler manifold possesses a Riemannian, a symplectic and a complex structure that are compatible in a sense we are going to explain now.

Recall that a *symplectic manifold* is a pair (M, ω) , where M is a $(2n)$ -dimensional smooth manifold and $\omega \in \Omega^2(M)$ is a closed 2-form on M such that ω is non-degenerate, i.e. $\omega_p^n \neq 0$ for all $p \in M$.

Definition 2.1.1.

- (i) An *almost complex structure* on an even-dimensional smooth manifold M is a complex structure J on the tangent bundle TM .
- (ii) Let M, J be as in (i) and $\omega \in \Omega^2(M)$ a non-degenerate 2-form on M . The 2-form ω is called *compatible* with J if the bilinear form $\langle \dots, \dots \rangle$ given by

$$\forall_{p \in M} \forall_{v, w \in T_p M} \langle v, w \rangle = \omega(v, Jw)$$

defines a Riemannian metric on M .

- (iii) An almost complex structure J on M as in (i) is called *integrable* if there exists an atlas \mathcal{A}_M on M such that

$$\forall u \in \mathcal{A}_M \quad \forall p \in \text{Domain}(u) \quad d_p u \circ J_p = J_0 \circ d_p u: T_p M \rightarrow \mathbb{R}^{2n},$$

where

$$J_0 = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}.$$

J is called *complex structure* for M .

- (iv) A *Kähler manifold* is a symplectic manifold (M, ω) with a complex structure J on M such that ω is compatible with J .

If one wants to show that a given almost complex structure is not integrable, it may be hard to disprove the condition (iii) of the last definition. But in [60], Newlander and Nirenberg proved their famous result that an almost complex structure J on a smooth manifold M is integrable if and only if $N_J \equiv 0$, where the *Nijenhuis tensor* N_J is defined as

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

for all vector fields X, Y on M .

For a time, it was not clear whether every symplectic manifold was not in fact Kählerian. Meanwhile, many examples of non-Kählerian symplectic manifolds are known. The first such was given by Thurston in 1976 – the so-called Kodaira-Thurston manifold, see [73].

The difficulty to prove non-existence of any Kähler structure is obvious. Nowadays, two easily verifiable necessary conditions for Kähler manifolds are known. First, we have the main theorem from the work [16] of Deligne, Griffiths, Morgan and Sullivan.

Theorem 2.1.2 ([16, p. 270]). *Compact Kähler manifolds are formal.* □

In order to prove that his manifold is not Kählerian, Thurston used another method. His manifold has first Betti number equal to three and the Hodge decomposition for Kähler manifolds implies that its odd degree Betti numbers have to be even, see e.g. [36, pp. 116 and 117]. This is even satisfied for every Hard Lefschetz manifold.

We say that a symplectic manifold (M^{2n}, ω) is *Lefschetz* if the homomorphism

$$\begin{aligned} L^k: \quad H^{n-k}(M, \mathbb{R}) &\longrightarrow H^{n+k}(M, \mathbb{R}) \\ [\alpha] &\longmapsto [\alpha \wedge \omega^k] \end{aligned}$$

is surjective for $k = n - 1$. If L^k is surjective for $k \in \{0, \dots, n - 1\}$, then (M, ω) is called *Hard Lefschetz*.

Note that for compact M the surjectivity of L^k implies its injectivity.

Obviously, the Lefschetz property depends on the choice of the symplectic form. It may be possible that a smooth manifold M possesses two symplectic forms ω_1, ω_2 such that (M, ω_1) is Lefschetz and (M, ω_2) not. But as mentioned above, the existence of such an ω_1 has the following consequence that is purely topological.

Theorem 2.1.3. *The odd degree Betti numbers of a Hard Lefschetz manifold are even.*

Proof. Let (M^{2n}, ω) be a symplectic manifold satisfying the Lefschetz property. We use the same idea as in [36, p. 123]. For each $i \in \{0, \dots, n-1\}$ one has a non-degenerated skew-symmetric bilinear form

$$\begin{array}{ccc} H^{2i+1}(M, \mathbb{R}) \times H^{2i+1}(M, \mathbb{R}) & \longrightarrow & \mathbb{R}, \\ ([\alpha], [\beta]) & \longmapsto & [\alpha \wedge \beta \wedge \omega^{n-2i-1}] \end{array}$$

i.e. $H^{2i+1}(M, \mathbb{R})$ must be even-dimensional. □

Obviously, this also proves the next corollary.

Corollary 2.1.4. *The first Betti number of a Lefschetz manifold is even.* □

Finally, the following shows that the statement of the last theorem holds for Kähler manifolds:

Theorem 2.1.5 ([36, p. 122]). *Compact Kähler manifolds are Hard Lefschetz.* □

The aim of the next sections is to prove the following theorems:

Theorem 2.1.6. *For all $m \in 2\mathbb{N}$, $m \geq 4$ and $b \in \mathbb{N}$, $b \geq 2$, there are compact m -dimensional symplectic manifolds with $b_1 = b$ which are non-formal.*

Theorem 2.1.7. *For all $m \in 2\mathbb{N}$, $m \geq 6$, there are compact m -dimensional symplectic manifolds with $b_1 = 1$ which are non-formal.*

Theorem 2.1.8. *For all $m \in 2\mathbb{N}$, $m \geq 8$, there are simply-connected compact m -dimensional symplectic manifolds which are non-formal.*

These three theorems and Theorem 1.3.1 imply:

Theorem 2.1.9. *Let $(m, b) \in 2\mathbb{N}_+ \times \mathbb{N}$. If there is a non-formal compact oriented m -dimensional manifold with $b_1 = b$, there is also a symplectic manifold with these properties.* □

2.2 Donaldson submanifolds

Our examples of non-formal symplectic manifolds will be constructed in a similar way as in the article [27] of Fernández and Muñoz. The examples will be Donaldson submanifolds of non-formal symplectic manifolds. Therefore, we quote in this section parts of [27].

Note, for the remainder of the chapter we denote the de Rham cohomology of a smooth manifold M by $H^*(M)$.

In [20] the following is proven: Let (M, ω) be a $2n$ -dimensional compact symplectic manifold with $[\omega] \in H^2(M)$ admitting a lift to an integral cohomology class. (Without loss of generality, the latter can always be assumed, see [34, Observation 4.3].) Then there exists $k_0 \in \mathbb{N}_+$ such that for each $k \in \mathbb{N}_+$ with $k \geq k_0$ there is a symplectic submanifold $j: Z \hookrightarrow M$ of dimension $2n - 2$ whose Poincaré dual satisfies $\text{PD}[Z] = k[\omega]$. Moreover, the map j is a homology $(n - 1)$ -equivalence in the following sense.

Let $f: M_1 \rightarrow M_2$ be a smooth map between smooth manifolds. f is called *homology s -equivalence*, $s \in \mathbb{N}$, if it induces isomorphisms $f^*: H^i(M_2) \rightarrow H^i(M_1)$ on cohomology for $i \leq s - 1$ and a monomorphism for $i = s$.

A symplectic submanifold $j: Z \hookrightarrow M$ as above is called *symplectic divisor* or *Donaldson submanifold*.

Concerning minimal models and formality in this context, we quote the following results. Part (i) resp. (ii) in the theorem coincides with Proposition 5.1 resp. Theorem 5.2 (i) in [27], where a proof is given.

Theorem 2.2.1 ([27]). *Let $f: M_1 \rightarrow M_2$ be a homology s -equivalence between connected smooth manifolds. Denote by $\rho_i: (\bigwedge V_i, d) \rightarrow (\Omega(M_i), d)$ the minimal models of M_i for $i = 1, 2$.*

(i) *There exist a morphism $F: (\bigwedge V_2^{\leq s}, d) \rightarrow (\bigwedge V_1^{\leq s}, d)$ of differential graded algebras such that $F: V_2^{\leq s} \rightarrow V_1^{\leq s}$ is an isomorphism, $F: V_2^s \rightarrow V_1^s$ is a monomorphism and $\rho_1^* \circ F^* = f^* \circ \rho_2^*$.*

(ii) *If M_2 is $(s - 1)$ -formal, then M_1 is $(s - 1)$ -formal.* □

Corollary 2.2.2 ([27, Theorem 5.2(ii)]). *Let M be a $2n$ -dimensional compact symplectic manifold and $j: Z \hookrightarrow M$ a Donaldson submanifold.*

Then for each $s \leq n - 2$, we have: If M is s -formal, then Z is s -formal.

In particular, Z is formal if M is $(n - 2)$ -formal. □

Next, we want to give a criterion for a Donaldson submanifold not to be formal.

Proposition 2.2.3. *Let M be a compact symplectic manifold of dimension $2n$, where $n \geq 3$. Using the notation from page 5, we suppose that there are*

cohomology classes $a_i = [\alpha_i] \in H^1(M)$, $1 \leq i \leq 3$, such that the (triple-)Massey product

$$\langle a_1, a_2, a_3 \rangle = [\alpha_1 \wedge \xi_{2,3} + \xi_{1,2} \wedge \alpha_3] \in \frac{H^2(M)}{a_1 \cup H^1(M) + H^1(M) \cup a_3}$$

is defined and does not vanish.

Then every Donaldson submanifold of M is not 1-formal.

Proof. Let $j: Z \hookrightarrow M$ be a Donaldson submanifold. Since $n \geq 3$, j is a homology 2-equivalence. This implies that the (triple-)Massey product

$$\langle j^*a_1, j^*a_2, j^*a_3 \rangle = [j^*\alpha_1 \wedge j^*\xi_{2,3} + j^*\xi_{1,2} \wedge j^*\alpha_3] \in \frac{H^2(Z)}{j^*a_1 \cup H^1(Z) + H^1(Z) \cup j^*a_3}$$

is defined and does not vanish. Now, Corollary 1.2.5 implies that Z is not 1-formal. \square

As an immediate consequence of the proposition we get:

Corollary 2.2.4. *Let Z_1, \dots, Z_k, M be compact symplectic manifolds and assume that $Z_i \hookrightarrow Z_{i+1}$ and $Z_k \hookrightarrow M$ are Donaldson submanifolds for $i = 1, \dots, k-1$. We suppose that there are cohomology classes $a_i = [\alpha_i] \in H^1(M)$, $1 \leq i \leq 3$, such that the (triple-)Massey product*

$$\langle a_1, a_2, a_3 \rangle = [\alpha_1 \wedge \xi_{2,3} + \xi_{1,2} \wedge \alpha_3] \in \frac{H^2(M)}{a_1 \cup H^1(M) + H^1(M) \cup a_3}$$

is defined and does not vanish.

If $\dim Z_1 \geq 4$, then Z_1 is not 1-formal. \square

The next lemma will be needed in the proof of Theorem 2.1.7. The proof is taken word by word from the proof of Formula (5) in [27]. Note that we denote the map $[\omega] \cup \dots: H^p(M) \rightarrow H^{p+2}(M)$ by $[\omega]: H^p(M) \rightarrow H^{p+2}(M)$.

Lemma 2.2.5. *Let (M, ω) be a $2n$ -dimensional compact symplectic manifold and $j: Z \hookrightarrow M$ a Donaldson submanifold.*

Then for each $p = 2(n-1) - i$, $0 \leq i \leq (n-2)$, there is a monomorphism

$$\frac{H^p(M)}{\ker([\omega]: H^p(M) \rightarrow H^{p+2}(M))} \longrightarrow H^p(Z).$$

Proof. The claim can be seen via Poincaré duality. Let $0 \leq i \leq (n-2)$, $p = 2(n-1) - i$ and $\alpha \in \Omega^p(M)$ be closed. Then we have

$$j^*[\alpha] = 0 \iff \forall_{b \in H^i(Z)} j^*[\alpha] \cup b = 0.$$

Since $i \leq (n - 2)$, we know that there is an isomorphism $j^*: H^i(M) \xrightarrow{\cong} H^i(Z)$, thus we can assume that for each $b \in H^i(Z)$ there is a closed i -form β on M with $[\beta|_Z] = j^*[\beta] = b$ and get

$$j^*[\alpha] \cup j^*[\beta] = \int_Z j^*\alpha \wedge j^*\beta = \int_M \alpha \wedge \beta \wedge k\omega,$$

since $[Z] = \text{PD}[k\omega]$ for $k \in \mathbb{N}_+$. Therefore, we have

$$j^*[\alpha] = 0 \iff \forall_{[\beta] \in H^i(M)} [\alpha \wedge \omega] \cup [\beta] = 0 \iff [\alpha \wedge \omega] = 0,$$

from where the lemma follows. \square

2.3 Known examples

In this and in the next but one section we make use of some basic results on nilmanifolds. Readers who are not familiar with this notion should read pages 22 up to 24 of Section 3.1 first.

2.3.1 The manifolds $M(p, q)$

The following examples are taken from [13].

Let R be a ring with 1. For $p \in \mathbb{N}_+$ let $H(1, p; R)$ be the set

$$\left\{ \begin{pmatrix} I_p & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, z \in R^p \wedge y \in R \right\}.$$

We write $H(1, p)$ for $H(1, p; \mathbb{R})$. Clearly, this is a nilpotent Lie group and the $2p + 1$ differential 1-forms

$$\alpha_i := dx_i, \beta := dy, \gamma_i := dz_i - x_i dy, \quad 1 \leq i \leq p,$$

form a basis of the left-invariant 1-forms. Obviously, we have $d\alpha_i = d\beta = 0$ and $d\gamma_i = -\alpha_i \wedge \beta$.

Further, let $q \in \mathbb{N}_+$. We set $G(p, q) := H(1, p) \times H(1, q)$. Again, this is a Lie group and analogous as above, we denote the $2p + 2q + 2$ forms which form a basis of the left-invariant 1-forms by

$$\alpha_1, \dots, \alpha_p, \beta, \gamma_1, \dots, \gamma_p, \tilde{\alpha}_1, \dots, \tilde{\alpha}_q, \tilde{\beta}, \tilde{\gamma}_1, \dots, \tilde{\gamma}_q.$$

An easy computation shows that the 2-form

$$\omega := \sum_{i=1}^p \alpha_i \wedge \gamma_i + \sum_{i=1}^q \tilde{\alpha}_i \wedge \tilde{\gamma}_i + \beta \wedge \tilde{\beta}$$

is a left-invariant symplectic form. Therefore $M(p, q) := G(p, q)/\Gamma(p, q)$, where $\Gamma(p, q) := H(1, p; \mathbb{Z}) \times H(1, q; \mathbb{Z})$, is a compact symplectic nilmanifold of dimension $2p + 2q + 2$. From Theorem 3.1.5 below we get the minimal model $\rho: (\mathcal{M}_{M(p, q)}, d) \rightarrow (\Omega(M(p, q)), d)$ as

$$\begin{aligned} \mathcal{M}_{M(p, q)} &= \bigwedge (a_1, \dots, a_p, b, c_1, \dots, c_p, \tilde{a}_1, \dots, \tilde{a}_q, \tilde{b}, \tilde{c}_1, \dots, \tilde{c}_q), \\ |a_i| &= |b| = |c_i| = |\tilde{a}_i| = |\tilde{b}| = |\tilde{c}_i| = 1, \\ da_i &= db = d\tilde{a}_i = d\tilde{b} = 0, \quad dc_i = -a_i b, \quad d\tilde{c}_i = -\tilde{a}_i \tilde{b}, \\ \rho(a_i) &= \alpha_i, \quad \rho(b) = \beta, \quad \rho(c_i) = \gamma_i, \quad \rho(\tilde{a}_i) = \tilde{\alpha}_i, \quad \rho(\tilde{b}) = \tilde{\beta}, \quad \rho(\tilde{c}_i) = \tilde{\gamma}_i. \end{aligned}$$

Therefore, we see $b_1(M(p, q)) = p + q + 2$.

Proposition 2.3.1 ([13]). *$M(p, q)$ is not formal.*

Proof. $\langle [\beta], [\alpha_i], [\alpha_i] \rangle$ is a non-vanishing Massey product. □

Using Theorem 3.1.5 again, one computes the first cohomology groups of $M(p, q)$ as

$$\begin{aligned} H^0(M(p, q)) &= \langle 1 \rangle, \\ H^1(M(p, q)) &= \langle [\alpha_i], [\beta], [\tilde{\alpha}_k], [\tilde{\beta}] \mid 1 \leq i \leq p, 1 \leq k \leq q \rangle, \\ H^2(M(p, q)) &= \langle [\alpha_i \wedge \gamma_j], [\alpha_i \wedge \tilde{\alpha}_k], [\alpha_i \wedge \tilde{\beta}], [\beta \wedge \gamma_j], [\beta \wedge \tilde{\alpha}_i], [\beta \wedge \tilde{\beta}], \\ &\quad [\tilde{\alpha}_k \wedge \tilde{\gamma}_l], [\tilde{\beta} \wedge \tilde{\gamma}_l] \mid 1 \leq i, j \leq p, 1 \leq k, l \leq q \rangle. \end{aligned}$$

2.3.2 The manifold $M_{8,0}$

Fernández and Muñoz constructed in [29] an 8-dimensional non-3-formal compact symplectic manifold $(M_{8,0}, \omega)$ with

$$\begin{aligned} b_0(M_{8,0}) &= b_8(M_{8,0}) = 1, \quad b_1(M_{8,0}) = b_7(M_{8,0}) = 0, \\ b_2(M_{8,0}) &= b_6(M_{8,0}) = 256, \quad b_3(M_{8,0}) = b_5(M_{8,0}) = 0, \quad b_4(M_{8,0}) = 269 \end{aligned} \quad (2.1)$$

as desingularisation of an orbifold. The latter is a \mathbb{Z}_3 -quotient of a nilmanifold. The non-formality is proved by regarding the G-Massey product $\langle [\vartheta]; [\tau_1], [\tau_2], [\tau_3] \rangle$ for certain closed 2-forms ϑ, τ_i on $M_{8,0}$: One has $\langle [\vartheta]; [\tau_1], [\tau_2], [\tau_3] \rangle = \lambda [\omega^4]$ for $\lambda \neq 0$. Clearly, $\lambda \omega^4$ is not exact, and since $b_3(M_{8,0}) = 0$, it follows from Definition 1.2.3 that this G-Massey product does not vanish.

2.4 Proofs

Proof of Theorem 2.1.6

Because products with finitely many copies of S^2 give the higher-dimensional examples, it is enough to prove that for every $b \geq 2$ there is a non-formal compact symplectic 4-manifold M with $b_1(M) = b$.

Examples for $b \in \{2, 3\}$ were given in [24]. We shall see them in the proofs of Theorems 2.5.5 and 3.6.2 below.

Now let $b \geq 4$ and choose $p, q \in \mathbb{N}_+$ such that $p + q + 2 = b$. Then $M(p, q)$ has dimension $2p + 2q + 2 \geq 6$ and is a non-formal compact symplectic nilmanifold with $b_1(M(p, q)) = b$ which satisfies the assumption of Corollary 2.2.4. Therefore, we get the required non-formal 4-manifold Z with $b_1(Z) = b_1(M(p, q)) = b$. \square

Proof of Theorem 2.1.7

Since direct products with finitely many copies of S^2 gives the higher-dimensional ones, it is enough to find a six-dimensional example. This will be constructed in Theorem 3.8.3.2 below.

But using the ideas from above, one can construct an eight dimensional example as follows:

Gompf has shown in [34] that there is a compact symplectic 4-manifold $M_{4,1}$ with $b_1(M_{4,1}) = 1$. By Proposition 1.1.7, $M_{12,1} := M_{8,0} \times M_{4,1}$ is a compact symplectic 12-manifold which is not 3-formal. Clearly, we have $b_1(M_{12,1}) = 1$. Denote the projections by $\pi: M_{12,1} \rightarrow M_{8,0}$, $\mathbf{p}: M_{12,1} \rightarrow M_{4,1}$ and the symplectic forms of $M_{8,0}$, $M_{4,1}$ and $M_{12,1}$ by ω, σ and $\Omega = \pi^*\omega + \mathbf{p}^*\sigma$. Let ϑ, τ_i be the 2-forms of Section 2.3.2. We mentioned $\langle [\vartheta]; [\tau_1], [\tau_2], [\tau_3] \rangle = \lambda [\omega^4] \neq 0$.

Let $j: Z_{10,1} \hookrightarrow M_{12,1}$ be a Donaldson submanifold. The 10-form

$$\Omega \wedge \lambda \pi^*\omega^4 = (\pi^*\omega + \mathbf{p}^*\sigma) \wedge \lambda \pi^*\omega^4 = \lambda \mathbf{p}^*\sigma \wedge \pi^*\omega^4$$

on $M_{12,1}$ does not represent the zero class in

$$H^{10}(M_{12,1}) \stackrel{(2.1)}{=} (\langle [\sigma^2] \rangle \otimes H^6(M_{8,0})) \oplus (H^2(M_{4,1}) \otimes H^8(M_{8,0})).$$

Therefore, we get from Lemma 2.2.5: $\lambda j^*\pi^*[\omega^4] \in H^8(Z_{10,1}) \setminus \{0\}$. From (2.1) we know $H^5(M_{8,0}) = 0$. Hence $\langle [j^*\pi^*\tau_k], [j^*\pi^*\vartheta], [j^*\pi^*\tau_l] \rangle = 0$ for $1 \leq k, l \leq 3$. So in the following G-Massey product there is no indeterminacy:

$$\langle [j^*\pi^*\vartheta]; [j^*\pi^*\tau_1], [j^*\pi^*\tau_2], [j^*\pi^*\tau_3] \rangle = \lambda j^*\pi^*[\omega^4] \neq 0$$

It follows that $Z_{10,1}$ is not formal. The fact that $\dim Z_{10,1} = 10$ and $b_1(Z_{10,1}) = 1$ is clear by the remarks in Section 2.2.

Now, let $\tilde{j}: Z_{8,1} \hookrightarrow Z_{10,1}$ be a Donaldson submanifold. Then the 10-form $j^*\Omega \wedge \lambda j^*\pi^*\omega^4$ on $Z_{10,1}$ does not represent the zero class in $H^{10}(Z_{10,1})$, for we have

$$\Omega^2 \wedge \pi^*\omega^4 = (\mathfrak{p}^*\sigma + \pi^*\omega) \wedge (\mathfrak{p}^*\sigma \wedge \pi^*\omega^4) = 2\mathfrak{p}^*\sigma^2 \wedge \pi^*\omega^4 \neq 0,$$

and by Lemma 2.2.5 we get $[j^*(\Omega \wedge \pi^*\omega^4)] \neq 0$.

Again we use Lemma 2.2.5 to see $\lambda \tilde{j}^*j^*\pi^*[\omega^4] \in H^8(Z_{8,1}) \setminus \{0\}$ and can prove similarly as for $Z_{10,1}$ that $Z_{8,1}$ is not formal. Moreover, $Z_{8,1}$ is 8-dimensional and has first Betti number equal to one. \square

Remark. A Donaldson submanifold $Z_{6,1}$ of the manifold $Z_{8,1}$ that we have constructed in the last proof is formal: From the 2-formality of $M_{12,1} = M_{8,0} \times M_{4,1}$ it follows that $Z_{6,1}$ is 2-formal and therefore formal by Theorem 1.1.6.

Proof of Theorem 2.1.8

$M_{8,0}$ is the eight dimensional example and the higher-dimensional examples are obtained by the taking product of $M_{8,0}$ with finitely many copies of S^2 . \square

2.5 Contact manifolds

We would like to end this chapter with a question that arises naturally once with have proved Theorem 2.1.9:

For which pairs (m, b) with m odd can we find a non-formal compact contact m -manifold with $b_1 = b$?

Recall that a *contact manifold* is a pair $(M, \xi = \ker \alpha)$, where M is a smooth $(2n+1)$ -manifold M and $\alpha \in \Omega^1(M)$ a 1-form with $\alpha_p \wedge (d\alpha)_p^n \neq 0$ for all $p \in M$. The hyperplane field ξ is called a *contact structure*; the 1-form α a *contact form on M* .

Theorem 2.5.1. *For each pair (m, b) with $m \geq 3$ odd and $b \geq 2$ there exists a non-formal compact contact m -manifold with $b_1 = b$.*

The remainder of the chapter is devoted to the proof of this theorem.

Our starting point is a non-formal symplectic manifold. Boothby and Wang proved that there is a contact manifold which fibres over it with fibre a circle.

Theorem 2.5.2 ([6, Theorem 3]). *If (M, ω) is a compact symplectic manifold whose symplectic form determines an integral cohomology class of M , then the principal circle bundle $\pi: E \rightarrow M$ with first Chern class $c_1(\pi) = [\omega] \in H^2(M, \mathbb{Z})$ admits a connection 1-form α such that $\pi^*\omega = d\alpha$ and α is a contact form on E .* \square

Let E, M be as in the last theorem. Since E is an S^1 -bundle over M , one can apply the Gysin sequence to obtain $b_1(E) = b_1(M)$. If $\dim M \geq 4$, we can even find a contact manifold which has the same fundamental group as M :

Corollary 2.5.3. *Let (M, ω) be a compact symplectic manifold of dimension $2n \geq 4$ whose symplectic form determines an integral cohomology class.*

Then there is a symplectic form ω' on $M \# \overline{\mathbb{C}P^n}$ determining an integer cohomology class, a compact contact manifold $(E, \ker \alpha')$, and a principal circle bundle $\pi: E \rightarrow M \# \overline{\mathbb{C}P^n}$ with first Chern class $c_1(\pi) = [\omega']$ such that the fundamental groups satisfy $\pi_1(E) = \pi_1(M \# \overline{\mathbb{C}P^n}) = \pi_1(M)$.

Proof. We use the same argumentation as in the proof of [34, Theorem 4.4]. After blowing up a point in M , we can obtain a manifold $M' := M \# \overline{\mathbb{C}P^n}$ with a symplectic form ω' such that $[\omega'] = [\omega] + \varepsilon e \in H^2(M') = H^2(M) \oplus H^2(\overline{\mathbb{C}P^n})$, where $\varepsilon \in \frac{1}{\mathbb{N}_+}$ and e is a generator of $H^*(\overline{\mathbb{C}P^n})$. Without loss of generality, we can assume that ω' determines an integral cohomology class and there is an embedded sphere $S \subset M' = M \# \overline{\mathbb{C}P^n}$ such that $\int_S \omega' = 1$. (Since $\int_S \omega'$ depends on the size of the ball removed from M in the blow-up, we may have to enlarge ω by an integer scale first.) Let $\pi: E \rightarrow M'$ with $c_1(\pi) = [\omega']$ as in Theorem 2.5.2. Then the restriction of the fibration π to S is the Hopf fibration, i.e. $\pi^{-1}(S) = S^3$ and the middle map in the following part of the homotopy sequence is an isomorphism:

$$\{0\} = \pi_2(\pi^{-1}(S)) \longrightarrow \pi_2(S) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(\pi^{-1}(S)) = \{1\}.$$

$\pi_2(S) \rightarrow \pi_1(S^1)$ is an isomorphism. From $S \subset M'$ we get in the following part of the homotopy sequence of the fibration π that the first map is surjective:

$$\pi_2(M') \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(E) \longrightarrow \pi_1(M') \longrightarrow \pi_0(S^1) = \{1\}.$$

This yields an isomorphism $\pi_*: \pi_1(E) \rightarrow \pi_1(M') = \pi_1(M)$. \square

Under certain conditions we can show that the contact manifold that we have just constructed is not formal.

Proposition 2.5.4. *Let (M, ω) be a compact symplectic manifold of dimension $2n \geq 4$ whose symplectic form determines an integral cohomology class. Further, suppose that there are cohomology classes $a_i \in H^1(M)$, $1 \leq i \leq 3$, such that $\langle a_1, a_2, a_3 \rangle$ is a non-vanishing Massey product in M .*

Then the manifold E of Corollary 2.5.3 is not formal.

Proof. Let $\pi: E \rightarrow M' := M \# \overline{\mathbb{C}P^n}$ be as in Corollary 2.5.3 and the non-vanishing Massey product $\langle a_1, a_2, a_3 \rangle$ be defined by a 2-form $\alpha_1 \cdot \xi_{2,3} + \xi_{1,2} \cdot \alpha_3$. (Here we use the notation from page 5.) We show:

$$\pi^*: H^1(M') \longrightarrow H^1(E) \text{ is an isomorphism,} \quad (2.2)$$

$$H^2(M) \cap \ker(\pi^*: H^2(M') \rightarrow H^2(E)) = \{0\}. \quad (2.3)$$

Then $\pi^* \alpha_1 \cdot \pi^* \xi_{2,3} + \pi^* \xi_{1,2} \cdot \pi^* \alpha_3$ defines the non-vanishing Massey product

$$\begin{aligned} \langle \pi^* a_1, \pi^* a_2, \pi^* a_3 \rangle &\in \frac{\pi^*(H^2(M'))}{\pi^* a_1 \cdot \pi^*(H^1(M')) + \pi^*(H^1(M')) \cdot \pi^* a_3} \\ &\subset \frac{H^2(E)}{\pi^* a_1 \cdot H^1(E) + H^1(E) \cdot \pi^* a_3}, \end{aligned}$$

so E is not formal.

[Assume $\langle \pi^* a_1, \pi^* a_2, \pi^* a_3 \rangle$ vanishes. Then for $j = 1, 2$ there exists a class $[\Xi_{j,j+1}] \in H^1(E)$ such that $0 = d\Xi_{j,j+1} = \pi^* \alpha_j \cdot \pi^* \alpha_{j+1}$. Property (2.2) implies the existence of $[\xi_{j,j+1}] \in H^1(M')$ with $0 = d\pi^* \xi_{j,j+1} = \pi^* \alpha_j \cdot \pi^* \alpha_{j+1}$ for $j = 1, 2$, i.e. $\alpha_j \cdot \alpha_{j+1}$ is exact by (2.3) and $\langle [\alpha_1], [\alpha_2], [\alpha_3] \rangle$ vanishes, which is a contradiction.]

It remains to show (2.2) and (2.3): Consider the Gysin sequence of π .

$$\{0\} \longrightarrow H^1(M') \xrightarrow{\pi^*} H^1(E) \longrightarrow H^0(M') \xrightarrow{[\omega'] \cup} H^2(M') \xrightarrow{\pi^*} H^2(E) \longrightarrow \dots \quad (2.4)$$

$[\omega'] \cup: H^0(M') \rightarrow H^2(M')$ is injective. Therefore, $\pi^*: H^1(M') \rightarrow H^1(E)$ is an isomorphism, i.e. (2.2) holds.

Further, we get $\ker(\pi^*: H^2(M') \rightarrow H^2(E)) \stackrel{(2.4)}{=} \mathbb{R}[\omega']$. Denote

$$\mathbf{pr}_2: H^2(M') = H^2(M) \oplus H^2(\overline{\mathbb{C}P^n}) \longrightarrow H^2(\overline{\mathbb{C}P^n})$$

the projection onto the second factor. Since ω' is the symplectic form of the blow-up of M , we have $\mathbf{pr}_2([\omega']) \neq 0$. But $\mathbf{pr}_2|_{H^2(M)} = 0$, so (2.3) follows. \square

Using the preparations that we have done, we are able to construct explicit non-formal contact manifolds.

Theorem 2.5.5. *For each $n \in \mathbb{N}$ with $n \geq 2$ and $b \in \{2, 3\}$ there exists a compact contact $(2n + 1)$ -manifold which is not formal.*

Proof. In [24] the following manifolds are studied. Let M_b , $b \in \{2, 3\}$ be the four dimensional nilmanifold with basis of left-invariant 1-forms $\{\alpha, \beta, \gamma, \eta\}$ such that

$$\begin{aligned} d\alpha &= d\beta = 0, \\ d\gamma &= \alpha \wedge \beta, \\ d\eta &= \begin{cases} \alpha \wedge \gamma & : b = 2 \\ 0 & : b = 3 \end{cases}. \end{aligned}$$

Then, $b_1(M_b) = b$, the 2-form $\alpha \wedge \eta + \beta \wedge \gamma$ is a symplectic form for M_b , and $\langle [\beta], [\beta], [\alpha] \rangle = -[\beta \wedge \gamma]$ is a non-vanishing Massey product. Again, one can assume that the symplectic form determines an integral cohomology class. The case $n = 2$ now follows from Proposition 2.5.4. For $n > 2$ consider the manifolds $M_b \times (S^2)^{n-2}$ instead of M_b . \square

Remark. The manifold M_3 in the last proof is the Kodaira-Thurston manifold that we mentioned at the beginning of this chapter.

Theorem 2.5.6. *For each $b \in \mathbb{N}$ with $b \geq 2$, there are non-formal compact contact manifolds of dimension 3 and 5 with first Betti number $b_1 = b$.*

Proof. By Theorem 1.3.1, we know that there is a compact oriented 3-manifold M with $b_1 = b \geq 2$ which is non-formal. By theorems of Martinet [52] and Geiges [33, Proposition 2] M and $M \times S^2$ admit contact structures. Further, it follows from Proposition 1.1.7 that $M \times S^2$ is not formal. \square

Now, Theorem 2.5.1 follows from Theorems 2.5.5, 2.5.6, Proposition 1.1.7 and the following result of Bourgeois:

Theorem 2.5.7 ([8]). *Let M be a compact contact manifold of dimension greater than or equal to three.*

Then $M \times T^2$ admits a contact structure. \square

Chapter 3

Solvmanifolds

In this chapter we want to study compact homogeneous spaces G/Γ , where G is a connected and simply-connected Lie group and Γ a discrete subgroup in G . It is well known that the existence of such a Γ implies the unimodularity of the Lie group G . Recall that a Lie group G is called *unimodular* if for all $X \in \mathfrak{g}$ holds $\text{tr ad } X = 0$, where \mathfrak{g} denotes the Lie algebra of G .

If we further demand G/Γ to be symplectic (when G is even-dimensional), a result of Chu [12] shows that G has to be solvable.

Therefore, we regard compact quotients of connected and simply-connected solvable Lie groups by discrete subgroups, so called *solvmanifolds*.

First, we recall the definition of nilpotent and solvable groups resp. Lie algebras.

- (i) Let G be group and denote its neutral element by e . We define the *derived series* $(D^{(k)}G)_{k \in \mathbb{N}}$, *descending series* $(G^{(k)})_{k \in \mathbb{N}}$ and *ascending series* $(G_{(k)})_{k \in \mathbb{N}}$ of subgroups in G inductively as follows:

$$\begin{aligned} D^{(0)}G &:= G^{(0)} := G, \\ D^{(k)}G &:= [D^{(k-1)}G, D^{(k-1)}G], \quad G^{(k)} := [G, G^{(k-1)}], \\ G_{(0)} &= \{e\}, \quad G_{(k)} := \{g \in G \mid [g, G] \subset G_{(k-1)}\}. \end{aligned}$$

G is called *nilpotent* if there exists $k_0 \in \mathbb{N}$ such that $G^{(k_0)} = \{e\}$.

G is called *solvable* if there exists $k_0 \in \mathbb{N}$ such that $D^{(k_0)}G = \{e\}$.

- (ii) Given a Lie algebra \mathfrak{g} , one defines the *derived*, *descending* and *ascending series* of subalgebras in \mathfrak{g} via

$$\begin{aligned} D^{(0)}\mathfrak{g} &:= \mathfrak{g}^{(0)} := \mathfrak{g}, \\ D^{(k)}\mathfrak{g} &:= [D^{(k-1)}\mathfrak{g}, D^{(k-1)}\mathfrak{g}], \quad \mathfrak{g}^{(k)} := [\mathfrak{g}, \mathfrak{g}^{(k-1)}], \\ \mathfrak{g}_{(0)} &= \{0\}, \quad \mathfrak{g}_{(k)} := \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset \mathfrak{g}_{(k-1)}\} \end{aligned}$$

and calls \mathfrak{g} *nilpotent* resp. *solvable* if its ascending resp. derived series becomes trivial for k_0 large enough.

We collect some properties in the following proposition. The parts which are not obvious can be found in [76, Section 3.18].

Proposition 3.0.1.

- (i) *The subgroups arising in the derived, descending and ascending series of a group are normal. Moreover, they are closed and simply-connected Lie subgroups in the case of a connected and simply-connected Lie group.*
- (ii) *The subalgebras arising in the derived, descending and ascending series of a Lie algebra are ideals.*
- (iii) *A Lie group is nilpotent resp. solvable if and only if its Lie algebra is nilpotent resp. solvable.* □

3.1 Nilmanifolds

We give a brief review of known results about a special kind of solvmanifolds, namely nilmanifolds. For the study non-formal symplectic manifolds, nilmanifolds form one of the best classes. On the one hand, the non-toral nilmanifolds introduce a geometrical complexity, while on the other hand their homotopy theory is still amenable to study. In particular, their minimal models are very easy to calculate and we shall see that each non-toral nilmanifold is non-formal.

A *nilmanifold* is a compact homogeneous space G/Γ , where G is a connected and simply-connected nilpotent Lie group and Γ a *lattice* in G , i.e. a discrete co-compact subgroup.

Example. Every lattice in the abelian Lie group \mathbb{R}^n is isomorphic to \mathbb{Z}^n . The corresponding nilmanifold is the n -dimensional torus. □

In contrast to arbitrary solvable Lie groups, there is an easy criterion for nilpotent ones which enables one to decide whether there is a lattice or not.

Recall that the exponential map $\exp: \mathfrak{g} \rightarrow G$ of a connected and simply-connected nilpotent Lie group is a diffeomorphism. We denote its inverse by $\log: G \rightarrow \mathfrak{g}$.

Theorem 3.1.1 ([66, Theorem 2.12]). *A simply-connected nilpotent Lie group G admits a lattice if and only if there exists a basis $\{X_1, \dots, X_n\}$ of the Lie algebra \mathfrak{g} of G such that the structure constants C_{ij}^k arising in the brackets*

$$[X_i, X_j] = \sum_k C_{ij}^k X_k$$

are rational numbers.

More precisely we have:

- (i) Let \mathfrak{g} have a basis with respect to which the structure constants are rational. Let $\mathfrak{g}_{\mathbb{Q}}$ be the vector space over \mathbb{Q} spanned by this basis. Then, if \mathcal{L} is any lattice of maximal rank in \mathfrak{g} contained in $\mathfrak{g}_{\mathbb{Q}}$, the group generated by $\exp(\mathcal{L})$ is a lattice in G .
- (ii) If Γ is a lattice in G , then the \mathbb{Z} -span of $\log(\Gamma)$ is a lattice \mathcal{L} of maximal rank in the vector space \mathfrak{g} such that the structure constants of \mathfrak{g} with respect to any basis contained in \mathcal{L} belong to \mathbb{Q} . \square

For a given lattice Γ in a connected and simply-connected nilpotent Lie group G , the subset $\log(\Gamma)$ need not to be an additive subgroup of the Lie algebra \mathfrak{g} .

Example. Consider the nilpotent Lie group $G := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$.

Its Lie algebra is $\mathfrak{g} := \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$, and the logarithm is given by

$$\log\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & x & z - \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The set of integer matrices contained in G forms a lattice Γ in G and

$$\log(\Gamma) = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}, (ab \equiv 0(2) \Rightarrow c \in \mathbb{Z}), (ab \equiv 1(2) \Rightarrow c \in \frac{1}{2}\mathbb{Z}) \right\}$$

is not a subgroup of \mathfrak{g} .

If Γ is a lattice such that $\log(\Gamma)$ is a subgroup of the Lie algebra, we call Γ a *lattice subgroup*.

Note that in the context of general Lie groups the name “lattice subgroup” has a different meaning, namely that G/Γ has a finite invariant measure. For nilpotent groups and discrete Γ , the latter is the same as to require that Γ is a lattice.

Theorem 3.1.2 ([14, Theorem 5.4.2]). *Let Γ be a lattice in a connected and simply-connected nilpotent Lie group.*

- (i) Γ contains a lattice subgroup of finite index.
- (ii) Γ is contained as a subgroup of finite index in a lattice subgroup. \square

For later uses, we quote the following two results.

Proposition 3.1.3 ([14, Lemma 5.1.4 (a)]). *Let G be a locally compact group, H a closed normal subgroup and Γ a discrete subgroup of G . Moreover, denote by $\pi: G \rightarrow G/H$ the natural map.*

If $\Gamma \cap H$ is a lattice in H , and Γ is a lattice in G , then $\pi(\Gamma)$ is a lattice in G/H and $\Gamma H = H\Gamma$ is a closed subgroup of G . \square

Theorem 3.1.4 ([14, p. 208]). *Let G be a connected and simply-connected nilpotent Lie group with lattice Γ and $k \in \mathbb{N}$.*

Then $\Gamma \cap D^{(k)}G$, $\Gamma \cap G^{(k)}$ resp. $\Gamma \cap G_{(k)}$ are lattices in $D^{(k)}G$, $G^{(k)}$ resp. $G_{(k)}$. Note, $G_{(1)}$ is the center $Z(G)$ of G . \square

We have seen that it is easy to decide if there is a lattice in a given connected and simply-connected nilpotent Lie group, i.e. if it induces a nilmanifold. Moreover, nilmanifolds have very nice properties which will be described now. Below, we shall see that these properties are not satisfied for general solvmanifolds.

Note that we can associate a DGA to each Lie algebra \mathfrak{g} as follows:

Let $\{X_1, \dots, X_n\}$ be a basis of \mathfrak{g} and denote by $\{x_1, \dots, x_n\}$ the dual basis of \mathfrak{g}^* . The *Chevalley-Eilenberg complex* of \mathfrak{g} is the differential graded algebra $(\wedge \mathfrak{g}^*, \delta)$ with δ given by

$$\delta(x_k) = - \sum_{i < j} C_{ij}^k x_i \wedge x_j,$$

where C_{ij}^k are the structure constants of $\{X_1, \dots, X_n\}$.

Theorem 3.1.5 ([61], [63, Theorem 2.1.3]). *Let G/Γ be a nilmanifold and denote by $\Omega_{l.i.}(G)$ the vector space of left-invariant differential forms on G .*

Then the natural inclusion $\Omega_{l.i.}(G) \rightarrow \Omega(G/\Gamma)$ induces an isomorphism on cohomology.

Moreover, the minimal model of G/Γ is isomorphic to the Chevalley-Eilenberg complex of the Lie algebra of G . \square

Corollary 3.1.6. *Any nilmanifold satisfies $b_1 \geq 2$.*

Proof. Let \mathfrak{g} be a nilpotent Lie algebra. By [78, Theorem 7.4.1] we have $H^1(\wedge \mathfrak{g}^*, \delta) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. By [18] any nilpotent Lie algebra \mathfrak{g} satisfies the inequality $\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \geq 2$ which then implies $b_1(\mathfrak{g}) \geq 2$. Hence the claim follows from the preceding theorem. \square

We now quote some results that show that it is easy to decide whether a nilmanifold is formal, Kählerian or Hard Lefschetz.

Theorem 3.1.7 ([39, Theorem 1]). *A nilmanifold is formal if and only if it is a torus. \square*

Theorem 3.1.8 ([63, Theorem 2.2.2]). *If a nilmanifold is Kählerian, then it is a torus. \square*

This theorem follows from Theorem 3.1.7. Another proof was given by Benson and Gordon in [4]. In fact they proved the following:

Theorem 3.1.9 ([4, pp. 514 et seq.]). *A symplectic non-toral nilmanifold is not Lefschetz.* \square

Corollary 3.1.10. *A symplectic nilmanifold is Hard Lefschetz if and only if it is a torus, independent of the special choice of the symplectic form.* \square

3.2 Solvmanifolds in general

A *solvmanifold* is a compact homogeneous space G/Γ , where G is a connected and simply-connected solvable Lie group and Γ a *lattice* in G , i.e. a discrete co-compact subgroup.

Remark. It is important to note that there is a more general notion of solvmanifold, namely a compact quotient of a connected and simply-connected solvable Lie group by a (possibly non-discrete) closed Lie subgroup (see [2]), but we are only considering solvmanifolds as in the last definition. Sometimes, such are called special solvmanifolds in the literature.

By [63, Theorem 2.3.11], a solvmanifold in our sense is necessary parallelisable. E.g. the Klein bottle (which can be written as compact homogeneous space of a three-dimensional connected and simply-connected solvable Lie group) is not a solvmanifold covered by our definition.

Obviously, every nilmanifold is also a solvmanifold. But most solvmanifolds are not diffeomorphic to nilmanifolds: Every connected and simply connected solvable Lie group is diffeomorphic to \mathbb{R}^m (see e.g. [76]), hence solvmanifolds are aspherical and their fundamental group is isomorphic to the considered lattice. Each lattice in a nilpotent Lie group must be nilpotent. But in general, lattices in solvable Lie group are not nilpotent and therefore the corresponding solvmanifolds are not nilmanifolds.

The fundamental group plays an important role in the study of solvmanifolds.

Theorem 3.2.1 ([66, Theorem 3.6]). *Let G_i/Γ_i be solvmanifolds for $i \in \{1, 2\}$ and $\varphi: \Gamma_1 \rightarrow \Gamma_2$ an isomorphism.*

Then there exists a diffeomorphism $\Phi: G_1 \rightarrow G_2$ such that

$$(i) \quad \Phi|_{\Gamma_1} = \varphi,$$

$$(ii) \quad \forall \gamma \in \Gamma_1 \forall p \in G_1 \quad \Phi(p\gamma) = \Phi(p)\varphi(\gamma). \quad \square$$

Corollary 3.2.2. *Two solvmanifolds with isomorphic fundamental groups are diffeomorphic.* \square

The study of solvmanifolds meets with noticeably greater obstacles than the study of nilmanifolds. Even the construction of solvmanifolds is considerably more difficult than is the case for nilmanifolds. The reason is that there is no simple criterion for the existence of a lattice in a connected and simply-connected solvable Lie group.

We shall quote some necessary criteria.

Proposition 3.2.3 ([53, Lemma 6.2]). *If a connected and simply-connected solvable Lie group admits a lattice then it is unimodular.* \square

Theorem 3.2.4 ([55],[63, Theorem 3.1.2]). *Let G/Γ be a solvmanifold that is not a nilmanifold and denote by N the nilradical of G .*

Then $\Gamma_N := \Gamma \cap N$ is a lattice in N , $\Gamma N = N\Gamma$ is a closed subgroup in G and $G/(N\Gamma)$ is a torus. Therefore, G/Γ can be naturally fibred over a non-trivial torus with a nilmanifold as fiber:

$$N/\Gamma_N = (N\Gamma)/\Gamma \longrightarrow G/\Gamma \longrightarrow G/(N\Gamma) = T^k$$

This bundle will be called the Mostow bundle. \square

Remark.

- (i) The structure group action of the Mostow bundle is given by left translations

$$N\Gamma/\Gamma_0 \times N\Gamma/\Gamma \longrightarrow N\Gamma/\Gamma,$$

where Γ_0 is the largest normal subgroup of Γ which is normal in $N\Gamma$. (A proof of the topological version of this fact can be found in [70, Theorem I.8.15]. The proof for the smooth category is analogous.)

- (ii) A non-toral nilmanifold G/Γ fibers over a non-trivial torus with fibre a nilmanifold, too, since $\Gamma \cap [G, G]$ resp. $\text{im}(\Gamma \rightarrow G/[G, G])$ are lattices in $[G, G]$ resp. $G/[G, G]$, see above.

In view of Theorem 3.2.4, we are interested in properties of the nilradical of a solvable Lie group. The following proposition was first proved in [56]. Since the paper is written in Russian and the author of this thesis does not speak Russian, it is possible that the proof below is the same as in [56].

Proposition 3.2.5. *Let G be a solvable Lie group and N its nilradical.*

Then $\dim N \geq \frac{1}{2} \dim G$.

Proof. Denote by $\mathfrak{n} \subset \mathfrak{g}$ the Lie algebras of $N \subset G$ and by $\mathfrak{n}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ their complexifications. Note that $\mathfrak{g}_{\mathbb{C}}$ is solvable with nilradical $\mathfrak{n}_{\mathbb{C}}$, so from [76, Corollary 3.8.4] it follows that $\mathfrak{n}_{\mathbb{C}} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid \text{ad}(X)|_{[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]} \text{ nilpotent}\}$. Therefore, since $\text{ad}: \mathfrak{g}_{\mathbb{C}} \rightarrow \text{Aut}([\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}])$ is a representation of $\mathfrak{g}_{\mathbb{C}}$ in $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$, by Lie's theorem

(e.g. [76, Theorem 3.7.3]) there exist $\lambda_1, \dots, \lambda_k \in \mathfrak{g}_{\mathbb{C}}^*$ such that $\mathfrak{n}_{\mathbb{C}} = \bigcap_{i=1}^k \ker \lambda_i$, where $k := \dim_{\mathbb{C}}[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$.

A straightforward calculation shows $\dim_{\mathbb{C}} \bigcap_{i=1}^k \ker \lambda_i \geq \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} - k$, so we have proven: $\dim_{\mathbb{C}} \mathfrak{n}_{\mathbb{C}} \geq \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} - \dim_{\mathbb{C}}[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$.

Because $\mathfrak{g}_{\mathbb{C}}$ is solvable, we get by [76, Corollary 3.8.4] that $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] \subset \mathfrak{n}_{\mathbb{C}}$ and hence $\dim_{\mathbb{C}} \mathfrak{n}_{\mathbb{C}} \geq \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} - \dim_{\mathbb{C}} \mathfrak{n}_{\mathbb{C}}$, i.e.

$$2 \dim_{\mathbb{C}} \mathfrak{n}_{\mathbb{C}} \geq \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}.$$

The proposition now follows from $\dim_{\mathbb{R}} \mathfrak{g} = \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}$ and $\dim_{\mathbb{R}} \mathfrak{n} = \dim_{\mathbb{C}} \mathfrak{n}_{\mathbb{C}}$. \square

In some cases, we will be able to apply the next theorem to the situation of Theorem 3.2.4. It then gives a sufficient condition for the Mostow bundle to be a principal bundle.

Theorem 3.2.6. *Let G be a connected and simply-connected solvable Lie group and Γ a lattice in G . Suppose that $\{e\} \neq H \subsetneq G$ is a closed normal abelian Lie subgroup of G with $H \subset N(\Gamma)$, the normalizer of Γ . (For example the latter is satisfied if H is central.) Assume further that $\Gamma_H := \Gamma \cap H$ is a lattice in H .*

Then $H/\Gamma_H = H\Gamma/\Gamma$ is a torus and

$$H/\Gamma_H \longrightarrow G/\Gamma \longrightarrow G/H\Gamma \tag{3.1}$$

is a principal torus bundle over a solvmanifold.

Proof. By assumption, H is a closed normal abelian subgroup of G and Γ_H is a lattice in H . We have for $h_1\gamma_1, h_2\gamma_2 \in H\Gamma$ with $h_i \in H, \gamma_i \in \Gamma$ the equivalence

$$(h_1\gamma_1)^{-1}(h_2\gamma_2) \in \Gamma \iff h_1^{-1}h_2 \in \Gamma_H,$$

i.e. $H/\Gamma_H = H\Gamma/\Gamma$. Therefore, Proposition 3.1.3 implies that (3.1) is a fibre bundle whose fibre is clearly a torus and its base a solvmanifold. The structure group action is given by the left translations

$$H\Gamma/\Gamma_0 \times H\Gamma/\Gamma \longrightarrow H\Gamma/\Gamma,$$

where Γ_0 is the largest normal subgroup of Γ which is normal in $H\Gamma$. (This can be seen analogous as in Remark (i) on page 26.) Since H is contained in $N(\Gamma) = \{g \in G \mid g\Gamma g^{-1} = \Gamma\}$, we have for each $h \in H$ and $\gamma, \gamma_0 \in \Gamma$

$$(h\gamma)\gamma_0(h\gamma)^{-1} = h\gamma\gamma_0\gamma^{-1}h^{-1} \in h\Gamma h^{-1} = \Gamma,$$

i.e. $\Gamma = \Gamma_0$ and the theorem follows. \square

We have seen that the Chevalley-Eilenberg complex associated to a nilmanifold is its minimal model. In this respect, arbitrary solvmanifolds differ in an essential way from nilmanifolds. However, in the special case of a solvmanifold which is the quotient of completely solvable Lie group, one has an access to the minimal model.

Definition 3.2.7. Let G be a Lie group with Lie algebra \mathfrak{g} .

- (i) G and \mathfrak{g} are called *completely solvable* if the linear map $\text{ad } X: \mathfrak{g} \rightarrow \mathfrak{g}$ has only real roots¹ for all $X \in \mathfrak{g}$.
- (ii) If G is simply-connected and $\exp: \mathfrak{g} \rightarrow G$ is a diffeomorphism, then G is called *exponential* and the inverse of \exp is denoted by $\log: G \rightarrow \mathfrak{g}$.

Remark. In the literature a connected and simply-connected Lie group is sometimes called exponential if the exponential map is surjective. This is weaker than our definition.

A nilpotent Lie group or algebra is completely solvable, and it is easy to see that completely solvable Lie groups or algebras are solvable. Moreover, the two propositions below show that simply-connected completely solvable Lie groups are exponential, and exponential Lie groups are solvable. Note that the second proposition is simply a reformulation of results of Saitô and Dixmier.

Proposition 3.2.8 ([64, Theorem 2.6.3]). *Any exponential Lie group is solvable.* \square

Proposition 3.2.9. *A connected and simply-connected solvable Lie group G with Lie algebra \mathfrak{g} is exponential if and only if the linear map $\text{ad } X: \mathfrak{g} \rightarrow \mathfrak{g}$ has no purely imaginary roots for all $X \in \mathfrak{g}$.*

Proof. Let G be a solvable Lie group. By [67, Théorème II.1], $\text{ad } X$ has no purely imaginary roots for all $X \in \mathfrak{g}$ if and only if the exponential map is surjective. If this is the case, [67, Théorème I.1] implies that the exponential map is even bijective. For solvable Lie groups, the statement “(1°) \Leftrightarrow (2°)” of [19, Théorème 3] says that this is equivalent to the exponential map being a diffeomorphism. \square

Let a lattice in a connected and simply-connected solvable Lie group be given. Then Theorem 3.2.4 stated that its intersection with the nilradical is a lattice in the nilradical. In the case of completely solvable Lie groups, we have an analogue for the commutator.

Proposition 3.2.10 ([35, Proposition 1]). *Let G be a connected and simply-connected completely solvable Lie group and Γ a lattice in G .*

Then $[\Gamma, \Gamma]$ is a lattice in $[G, G]$. In particular, $\Gamma \cap [G, G]$ is a lattice in $[G, G]$. \square

We formulate the result that enables us to compute the minimal model of solvmanifolds which are built by dividing a lattice out of a completely solvable group. The main part of the next theorem is due to Hattori [44].

¹By a root of a linear map, we mean a (possibly non-real) root of the characteristic polynomial.

Theorem 3.2.11. *Let G/Γ be a solvmanifold. Denote by $(\bigwedge \mathfrak{g}^*, \delta)$ the Chevalley-Eilenberg complex of G and recall that \mathfrak{g}^* is the set of left-invariant differential 1-forms on G . Then the following holds:*

- (i) *The natural inclusion $(\bigwedge \mathfrak{g}^*, \delta) \rightarrow (\Omega(G/\Gamma), d)$ induces an injection on cohomology.*
- (ii) *If G is completely solvable, then the inclusion in (i) is a quasi-isomorphism, i.e. it induces an isomorphism on cohomology. Therefore, the minimal model $\mathcal{M}_{G/\Gamma}$ is isomorphic to the minimal model of the Chevalley-Eilenberg complex.*
- (iii) *If $\text{Ad}(\Gamma)$ and $\text{Ad}(G)$ have the same Zariski closures², then the inclusion in (i) is a quasi-isomorphism. \square*

Proof. (i) is [63, Theorem 3.2.10] and (iii) is [66, Corollary 7.29].

ad (ii): Denote the mentioned inclusion by $i: (\bigwedge \mathfrak{g}^*, \delta) \rightarrow (\Omega(G/\Gamma), d)$. By Hattori's Theorem (see [63, p. 77]), this is a quasi-isomorphism. It remains to show that the minimal model $\rho: (\mathcal{M}_{CE}, \delta_{CE}) \rightarrow (\bigwedge \mathfrak{g}^*, \delta)$ of $(\bigwedge \mathfrak{g}^*, \delta)$ is the minimal model of $(\Omega(G/\Gamma), d)$. Since the minimal model is unique and the map $i \circ \rho: (\mathcal{M}_{CE}, \delta_{CE}) \rightarrow (\Omega(G/\Gamma), d)$ is a quasi-isomorphism, this is obvious. \square

There are examples where the inclusion in (i) in the last theorem is not a quasi-isomorphism: Consider the Lie group G which is \mathbb{R}^3 as a manifold and whose Lie group structure is given by

$$(s, a, b) \cdot (t, x, y) = (s + t, \cos(2\pi t) a - \sin(2\pi t) b + x, \sin(2\pi t) a + \cos(2\pi t) b + y).$$

G is not completely solvable and one calculates for its Lie algebra $b_1(\mathfrak{g}) = 1$. G contains the abelian lattice $\Gamma := \mathbb{Z}^3$ and G/Γ is the 3-torus which has $b_1 = 3$.

We have seen in the last section that the first Betti number of a nilmanifold is greater than or equal to two. For arbitrary solvmanifolds this is no longer true. Below, we shall see various examples of solvmanifolds with $b_1 = 1$. The following corollary shows that $b_1 = 0$ cannot arise.

Corollary 3.2.12. *Any solvmanifold satisfies $b_1 \geq 1$.*

Proof. Let \mathfrak{g} be a solvable Lie algebra. As in the nilpotent case we have $b_1(\bigwedge \mathfrak{g}^*, \delta) = \dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, and $\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \geq 1$ by solvability. The claim now follows from Theorem 3.2.11 (i). \square

²A basis for the Zariski topology on $\text{GL}(m, \mathbb{C})$ is given by the sets

$$U_p := \text{GL}(m, \mathbb{C}) \setminus p^{-1}(\{0\}),$$

where $p: \text{GL}(m, \mathbb{C}) \cong \mathbb{C}^{(m^2)} \rightarrow \mathbb{C}$ ranges over polynomials.

To end this section, we shortly discuss the existence problem for Kähler structures on solvmanifolds. The only Kählerian nilmanifolds are tori, but in the general context we have the hyperelliptic surfaces, which are non-toral Kählerian solvmanifolds, see Section 3.6 below. ([63, Theorem 3.4.1] states the only Kählerian solvmanifolds in dimension four are tori. This is not correct, as first noted by Hasegawa in [42].) Benson and Gordon [5] conjectured in 1990 that the existence of a Kähler structure on a solvmanifold G/Γ with G completely solvable forces G/Γ to be toral and this is true. In fact, Hasegawa proved in the first half of this decade the following:

Theorem 3.2.13 ([42]). *A solvmanifold G/Γ is Kählerian if and only if it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus.*

If G is completely solvable, then G/Γ is Kählerian if and only if it is a complex torus. \square

In later sections we shall see that neither the Hard Lefschetz property nor formality is sufficient for an even-dimensional solvmanifold to be Kählerian.

3.3 Semidirect products

In later sections we shall try to examine low-dimensional solvmanifolds. Concerning this, a first step is to use the known classification of the (connected and simply-connected) low-dimensional solvable Lie groups. Most of them have the structure of semidirect products. In order to define this notion, we recall the construction of the Lie group structure of the group of Lie group automorphisms of a simply-connected Lie group in the following theorem. It collects results that can be found in [77, pp. 117 et seq.].

Theorem 3.3.1.

(i) *Let $\mathfrak{h} = (|\mathfrak{h}| = \mathbb{R}^h, [\dots, \dots])$ be an h -dimensional Lie algebra. Then the set $A(\mathfrak{h})$ of Lie algebra isomorphisms of \mathfrak{h} is a closed Lie subgroup of the automorphism group $\text{Aut}(|\mathfrak{h}|)$ of the h -dimensional vector space $|\mathfrak{h}|$. The Lie algebra of $A(\mathfrak{h})$ is*

$$\mathfrak{d}(\mathfrak{h}) = \{\varphi \in \text{End}(|\mathfrak{h}|) \mid \varphi \text{ derivation with respect to } [\dots, \dots]\}.$$

(ii) *Let H be a connected and simply-connected Lie group with neutral element e and Lie algebra \mathfrak{h} . The Lie group structure of $A(H)$, the group of Lie group automorphisms of H , is given by the following group isomorphism:*

$$A(H) \longrightarrow A(\mathfrak{h}), \quad f \longmapsto d_e f.$$

Moreover, if H is exponential, its inverse is the map

$$\mathbf{A}(\mathfrak{h}) \longrightarrow \mathbf{A}(H), \quad \varphi \longmapsto \exp^H \circ \varphi \circ \log^H.$$

□

For given (Lie) groups G, H and a (smooth) action $\mu: G \times H \rightarrow H$ by (Lie) group automorphisms, one defines the *semidirect product of G and H via μ* as the (Lie) group $G \rtimes_{\mu} H$ with underlying set (manifold) $G \times H$ and group structure defined as follows:

$$\forall_{(g_1, h_1), (g_2, h_2) \in G \times H} (g_1, h_1)(g_2, h_2) = (g_1 g_2, \mu(g_2^{-1}, h_1) h_2)$$

Note that for $(g, h) \in G \rtimes_{\mu} H$ we have $(g, h)^{-1} = (g^{-1}, \mu(g, h^{-1}))$.

If the action μ is trivial, i.e. $\forall_{g \in G, h \in H} \mu(g, h) = h$, one obtains the ordinary direct product. In the case of Lie groups G and H , the exponential map $\exp^{G \times H}$ is known to be the direct product of \exp^G and \exp^H . If the action is not trivial, the situation becomes a little more complicated:

Theorem 3.3.2. *Let G, H be connected Lie groups and $\mu: G \times H \rightarrow H$ a smooth action by Lie group automorphisms. Denote the Lie algebras of G and H by \mathfrak{g} and \mathfrak{h} and let $\phi := (d_{e_G} \mu_1): \mathfrak{g} \rightarrow \mathfrak{d}(\mathfrak{h})$, where $\mu_1: G \rightarrow \mathbf{A}(\mathfrak{h})$ is given by $\mu_1(g) = d_{e_H} \mu(g, \dots) = \text{Ad}_g^{G \rtimes_{\mu} H}$.*

- (i) *The Lie algebra of $G \rtimes_{\mu} H$ is $\mathfrak{g} \rtimes_{\phi} \mathfrak{h}$. This Lie algebra is called semidirect product of \mathfrak{g} and \mathfrak{h} via ϕ . Its underlying vector space is $\mathfrak{g} \times \mathfrak{h}$ and the bracket for $(X_1, Y_1), (X_2, Y_2) \in \mathfrak{g} \times \mathfrak{h}$ is given by*

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2]_{\mathfrak{g}}, [Y_1, Y_2]_{\mathfrak{h}} + \phi(X_1)(Y_2) - \phi(X_2)(Y_1)).$$

In the sequel we shall identify $X \equiv (X, 0)$ and $Y \equiv (0, Y)$.

- (ii) *For $(X, Y) \in \mathfrak{g} \rtimes_{\phi} \mathfrak{h}$ one has $\exp^{G \rtimes_{\mu} H}((X, Y)) = (\exp^G(X), \gamma(1))$, where $\gamma: \mathbb{R} \rightarrow H$ is the solution of*

$$\dot{\gamma}(t) = (d_{e_H} R_{\gamma(t)}) (\exp^{\mathbf{A}(\mathfrak{h})}(-t \text{ad}(X)|_{\mathfrak{h}})(Y)), \quad \gamma(0) = e_H.$$

Here R_a denotes the right translation by an element $a \in H$.

Proof. The proof of (i) can be found in [76]. We give a proof of (ii). Given a Lie group homomorphism f between Lie groups, we denote its differential at the neutral element by f_* .

For $(g_0, h_0), (g, h) \in G \rtimes_{\mu} H$ we have $R_{(g_0, h_0)}(g, h) = (R_{g_0}(g), R_{h_0}(\mu(g_0^{-1}, h)))$, and this yields for $(X, Y) \in \mathfrak{g} \rtimes_{\phi} \mathfrak{h}$

$$(R_{(g_0, h_0)})_*((X, Y)) = ((R_{g_0})_*(X), (R_{h_0})_*(\mu_1(g_0^{-1})(Y))).$$

Since $(\gamma_1(t), \gamma_2(t)) := \exp^{G \times_\mu H}(t(X, Y))$ is the integral curve through the identity of both the right- and left-invariant vector fields associated to (X, Y) , the last equation implies that $(\gamma_1(t), \gamma_2(t))$ is the solution of the following differential equations:

$$\gamma_1(0) = e_G, \quad \dot{\gamma}_1(t) = (R_{\gamma_1(t)})_*(X), \quad (3.2)$$

$$\gamma_2(0) = e_H, \quad \dot{\gamma}_2(t) = (R_{\gamma_2(t)})_*(\mu_1(\gamma_1(-t))(Y)). \quad (3.3)$$

$\gamma_1(t) = \exp^G(tX)$ is the solution of (3.2), and this implies

$$\mu_1(\gamma_1(-t)) = \text{Ad}_{\gamma_1(-t)}^{G \times_\mu H}|_{\mathfrak{h}} = \exp^{A(\mathfrak{h})}(-t \text{ad}(X)|_{\mathfrak{h}}),$$

i.e. (3.3) is equivalent to $\gamma_2(0) = e_H$, $\dot{\gamma}_2(t) = (R_{\gamma_2(t)})_*(\exp^{A(\mathfrak{h})}(-t \text{ad}(X)|_{\mathfrak{h}})(Y))$. So the theorem is proven. \square

A connected and simply-connected solvable Lie group G with nilradical N is called *almost nilpotent* if it can be written as $G = \mathbb{R} \times_\mu N$. Moreover, if N is abelian, i.e. $N = \mathbb{R}^n$, then G is called *almost abelian*.

Let $G = \mathbb{R} \times_\mu N$ be an almost nilpotent Lie group. Since N has codimension one in G , we can consider μ as a one-parameter group $\mathbb{R} \rightarrow A(N)$. By Theorem 3.3.1, there exists $\varphi \in \mathfrak{d}(\mathfrak{n})$ with

$$\forall t \in \mathbb{R} \quad \mu(t) = \exp^N \circ \exp^{\text{Aut}(|\mathfrak{n}|)}(t\varphi) \circ \log^N.$$

Choosing a basis of $|\mathfrak{n}|$, we can identify $\text{Aut}(|\mathfrak{n}|)$ with a subset of $\mathfrak{gl}(n, \mathbb{R})$ and get

$$\forall t \in \mathbb{R} \quad d_e(\mu(t)) \in \exp^{\text{GL}(n, \mathbb{R})}(\mathfrak{gl}(n, \mathbb{R})).$$

Note, if N is abelian, the exponential map $\exp^N: \mathfrak{n} \rightarrow N$ is the identity. These considerations make it interesting to examine the image of $\exp^{\text{GL}(n, \mathbb{R})}$.

Theorem 3.3.3 ([62, Theorem 6]). *M is an element of $\exp^{\text{GL}(n, \mathbb{R})}(\mathfrak{gl}(n, \mathbb{R}))$ if and only if the real Jordan form of M contains in the form of pairs the blocks belonging to real negative eigenvalues λ_i^- , whenever there exist real negative eigenvalues λ_i^- of M . I.e. the block belonging to such a λ_i^- is of the following form*

$$\bigoplus_{j=1}^{n_i} \begin{pmatrix} J_{n_{ij}} & 0 \\ 0 & J_{n_{ij}} \end{pmatrix}$$

with

$$J_{n_{ij}} = \begin{pmatrix} \lambda_i^- & 1 & & 0 \\ & \lambda_i^- & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i^- \end{pmatrix} \in M(n_{ij}, n_{ij}; \mathbb{R}).$$

\square

We are now going to derive some facts that follow from the existence of a lattice in an almost nilpotent Lie group.

Theorem 3.3.4 ([74]). *Let $G = \mathbb{R} \ltimes_{\mu} N$ be an almost nilpotent and completely solvable Lie group containing a lattice Γ .*

Then there is a one-parameter group $\nu: \mathbb{R} \rightarrow \mathbf{A}(N)$ such that $\nu(k)$ preserves the lattice $\Gamma_N := \Gamma \cap N$ for all $k \in \mathbb{Z}$. Γ is isomorphic to $\mathbb{Z} \ltimes_{\nu} \Gamma_N$ and G/Γ is diffeomorphic to $(\mathbb{R} \ltimes_{\nu} N)/(\mathbb{Z} \ltimes_{\nu} \Gamma_N)$.

Moreover, there are $t_1 \in \mathbb{R} \setminus \{0\}$ and an inner automorphism $I_{n_1} \in \mathbf{A}(N)$ such that $\nu(1) = \mu(t_1) \circ I_{n_1}$.

Proof. We know that Γ_N is a lattice in N and $\text{im}(\Gamma \rightarrow G/N) \cong \Gamma/\Gamma_N$ is a lattice in $G/N \cong \mathbb{R}$. Therefore, $\Gamma/\Gamma_N \cong \mathbb{Z}$ is free, and by Proposition C.2 the following exact sequence is split:

$$\{1\} \longrightarrow \Gamma_N \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow \{0\},$$

i.e. there is a group-theoretic section $s: \mathbb{Z} \rightarrow \Gamma$. [67, Théorème II.5] states that a group homomorphism from a lattice of completely solvable Lie group into another completely solvable Lie group uniquely extends to a Lie group homomorphism of the Lie groups. Hence, s extends uniquely to a one-parameter group $s: \mathbb{R} \rightarrow G$. Therefore,

$$\nu: \mathbb{R} \longrightarrow \mathbf{A}(N), \quad \nu(t)(n) = s(t) \cdot n \cdot s(t)^{-1},$$

is a one-parameter group with $\forall_{k \in \mathbb{Z}} \nu(k)(\Gamma_N) = \Gamma_N$, the lattice Γ is isomorphic to $\mathbb{Z} \ltimes_{\nu} \Gamma_N$ by Proposition C.7 and G/Γ is diffeomorphic to $(\mathbb{R} \ltimes_{\nu} N)/(\mathbb{Z} \ltimes_{\nu} \Gamma_N)$.

Let $\gamma_1 := s(1) \in (\Gamma \setminus \Gamma_N) \subset \mathbb{R} \ltimes_{\mu} N$. There are unique $t_1 \in \mathbb{R} \setminus \{0\}$, $n_1 \in N$ with $\gamma_1 = t_1 \cdot n_1$, where we identify $t_1 \equiv (t_1, e_N) \in G$ and $n_1 \equiv (0, n_1) \in G$. Since $G = \mathbb{R} \ltimes_{\nu} N$ and $G = \mathbb{R} \ltimes_{\mu} N$ with the same normal subgroup N of G , one has for all $n \in N$

$$\nu(1)(n) = \gamma_1 \cdot n \cdot \gamma_1^{-1} = t_1 \cdot n_1 \cdot n \cdot n_1^{-1} \cdot t_1^{-1} = \mu(t_1)(n_1 \cdot n \cdot n_1^{-1}) = \mu(t_1)(I_{n_1}(n)),$$

from where the theorem follows. \square

Corollary 3.3.5. *Let $G = \mathbb{R} \ltimes_{\mu} N$ be an almost nilpotent (not necessary completely solvable) Lie group containing a lattice Γ . Again, denote by $\Gamma_N := \Gamma \cap N$ the induced lattice in the nilradical of G .*

Then there exist $t_1 \in \mathbb{R} \setminus \{0\}$, a group homomorphism $\nu: \mathbb{Z} \rightarrow \text{Aut}(\Gamma_N)$, and an inner automorphism I_{n_1} of N such that $\Gamma \cong \mathbb{Z} \ltimes_{\nu} \Gamma_N$ and $\nu(1) = \mu(t_1) \circ I_{n_1}$.

If G is almost abelian, then a basis transformation yields $\Gamma \cong t_1 \mathbb{Z} \ltimes_{\mu|_{\mathbb{Z}^n}} \mathbb{Z}^n$.

Proof. We argue as in the last proof. But we do not use [67, Théorème 5] and get only a group homomorphism $\nu: \mathbb{Z} \rightarrow \text{Aut}(\Gamma_N)$ (defined on \mathbb{Z} instead of \mathbb{R}). For general N , the calculation at the end of the proof implies the claim.

Since an abelian group has only one inner automorphism, in the almost abelian case this yields $\nu(1) = \mu(t_1)|_{\Gamma_N}$, so ν can be extended to $\nu: \mathbb{R} \rightarrow \mathbf{A}(\mathbb{R}^n)$ via $\nu(t) := \mu(t \cdot t_1)$. Further, by Corollary 3.2.2, we have $\Gamma_N \cong \mathbb{Z}^n$. \square

Hence we have seen, that the existence of a lattice in an almost nilpotent Lie group implies that a certain Lie group automorphism must preserve a lattice in the (nilpotent) nilradical. The next theorem deals with such automorphisms.

Theorem 3.3.6. *Let N be a connected and simply-connected nilpotent Lie group with Lie algebra \mathfrak{n} , $f_* \in \mathbf{A}(\mathfrak{n})$, and $f := \exp^N \circ f_* \circ \log^N \in \mathbf{A}(N)$, i.e. $d_e f = f_*$. Assume that f preserves a lattice Γ in N .*

Then there exists a basis \mathfrak{X} of \mathfrak{n} such that $M_{\mathfrak{X}}(f_) \in \mathrm{GL}(n, \mathbb{Z})$, where $M_{\mathfrak{X}}(f_*)$ denotes the matrix of f_* with respect to \mathfrak{X} .*

Moreover, if there are a one-parameter group $\mu: \mathbb{R} \rightarrow \mathbf{A}(N)$ and $t_0 \neq 0$ such that $\mu(t_0) = f$, i.e. $d_e(\mu(t_0)) = f_$, then $\det(d_e(\mu(\dots))) \equiv 1$.*

Proof. By Theorem 3.1.1 (ii),

$$\mathcal{L} := \langle \log^N(\Gamma) \rangle_{\mathbb{Z}} = \left\{ \sum_{i=1}^m k_i V_i \mid m \in \mathbb{N}_+, k_i \in \mathbb{Z}, V_i \in \log^N(\Gamma) \right\}$$

is a lattice in \mathfrak{n} . Therefore, there exists a basis $\mathfrak{X} = \{X_1, \dots, X_n\}$ of \mathfrak{n} such that $\mathcal{L} = \langle \mathfrak{X} \rangle_{\mathbb{Z}}$.

Since $f(\Gamma) \subset \Gamma$, we have $f_*(\log^N(\Gamma)) \subset \log^N(\Gamma)$. This implies $f_*(\mathcal{L}) \subset \mathcal{L}$ and hence, $M_{\mathfrak{X}}(f_*) \in \mathrm{GL}(n, \mathbb{Z})$.

Further, if $\mu(t_0) = f$ with μ , $t_0 \neq 0$ as in the statement of the theorem, then the map $\Delta := \det \circ d_e(\mu(\dots)): (\mathbb{R}, +) \rightarrow (\mathbb{R} \setminus \{0\}, \cdot)$ is a continuous group homomorphism with $\Delta(0) = 1$ and $\Delta(t_0) = \pm 1$, i.e. $\Delta \equiv 1$. \square

Remark. The basis \mathfrak{X} in the last theorem has rational structure constants.

Obviously, a one-parameter group μ in the automorphism group of an abelian Lie group with $\mu(t_0)$ integer valued for $t_0 \neq 0$ defines a lattice in $\mathbb{R} \times_{\mu} \mathbb{R}^n$. It is easy to compute the first Betti number of the corresponding solvmanifold, as the next proposition will show. Before stating it, we mention that the situation becomes more complicated in the case of a non-abelian and nilpotent group N .

Let a one-parameter group $\mu: \mathbb{R} \rightarrow \mathbf{A}(N)$ be given and $t_0 \neq 0$ such that $d_e(\mu(t_0))$ is an integer matrix with respect to a basis \mathfrak{X} of the Lie algebra \mathfrak{n} of N . In general, this does not enable us to define a lattice in $\mathbb{R} \times_{\mu} N$. But if $\Gamma_N := \exp^N(\langle \mathfrak{X} \rangle_{\mathbb{Z}})$ is a lattice in N , i.e. Γ_N is a lattice group, then this is possible.

Proposition 3.3.7. *Let $\mu: \mathbb{R} \rightarrow \mathrm{SL}(n, \mathbb{R})$ be a one-parameter group such that $\mu(1) = (m_{ij})_{i,j} \in \mathrm{SL}(n, \mathbb{Z})$.*

Then $M := (\mathbb{R} \times_{\mu} \mathbb{R}^n) / (\mathbb{Z} \times_{\mu} \mathbb{Z}^n)$ is a solvmanifold with

$$\pi_1(M) = \langle e_0, e_1, \dots, e_n \mid \begin{array}{l} \forall_{i \in \{1, \dots, n\}} e_0 e_i e_0^{-1} = e_1^{m_{1i}} \dots e_n^{m_{ni}} \\ \forall_{i, j \in \{1, \dots, n\}} [e_i, e_j] = 1 \end{array} \rangle$$

and $b_1(M) = n + 1 - \text{rank}(\mu(1) - \text{id})$.

Proof. The statement about the fundamental group is clear. Therefore, we get

$$H_1(M, \mathbb{Z}) = \langle e_0, e_1, \dots, e_n \mid \begin{array}{l} \forall_{i \in \{1, \dots, n\}} e_1^{m_{1i}} \dots e_i^{m_{ii}-1} \dots e_n^{m_{ni}} = 1 \\ \forall_{i, j \in \{0, \dots, n\}} [e_i, e_j] = 1 \end{array} \rangle$$

and this group is the abelianisation of

$$\mathbb{Z} \oplus \langle e_1, \dots, e_n \mid \forall_{i \in \{1, \dots, n\}} e_1^{m_{1i}} \dots e_i^{m_{ii}-1} \dots e_n^{m_{ni}} = 1 \rangle.$$

Now, the proof of the theorem about finitely generated abelian groups (see e.g. [7]) shows $H_1(M, \mathbb{Z}) = \mathbb{Z}^{n-k+1} \oplus \bigoplus_{i=1}^k \mathbb{Z}_{d_i}$, where $d_1, \dots, d_k \in \mathbb{N}_+$ denote the elementary divisors of $\mu(1) - \text{id}$. The proposition follows. \square

We finally mention a result of Gorbatsevich. In view of Theorem 3.2.11 (iii), it enables us to compute the minimal model of a wide class of solvmanifolds which are discrete quotients of almost abelian Lie groups.

Theorem 3.3.8 ([35, Theorem 4]). *Let $\mu: \mathbb{R} \rightarrow \text{SL}(n, \mathbb{R})$ be a one-parameter group such that $\mu(1) = \exp^{\text{SL}(n, \mathbb{R})}(\dot{\mu}(0)) \in \text{SL}(n, \mathbb{Z})$. Denote by $\lambda_1, \dots, \lambda_n$ the (possibly not pairwise different) roots of $\dot{\mu}(0)$. Then $\Gamma := (\mathbb{Z} \times_{\mu} \mathbb{Z}^n)$ is a lattice in $G := (\mathbb{R} \times_{\mu} \mathbb{R}^n)$.*

The Zariski closures of $\text{Ad}(\Gamma)$ and $\text{Ad}(G)$ coincide if and only if the number πi is not representable as a linear combination of the numbers λ_k with rational coefficients. \square

3.4 Semisimple splittings

In distinction from the nilpotent case, criteria for the existence of a lattice in connected and simply-connected solvable Lie groups have rather cumbersome formulations. The criterion that we present is due to Auslander [2] and makes use of the concept of semisimple splitting.

Let G be a connected and simply-connected Lie group. We call a connected and simply-connected solvable Lie group $G_s = T \times_{\nu_s} N_s$ a *semisimple splitting* for G if the following hold:

- (i) N_s is the nilradical of G_s – the so called *nilshadow* of G – and $T \cong \mathbb{R}^k$ for $k = \dim G_s - \dim N_s$,

- (ii) T acts on N_s via ν_s by semisimple automorphisms,
- (iii) G is a closed normal subgroup of G_s and $G_s = T \ltimes_{\varpi} G$,
- (iv) $N_s = Z_{N_s}(T) \cdot (N_s \cap G)$, where $Z_{N_s}(T)$ denotes the centralizer of T in N_s .

This definition then implies (see e.g. [15, Lemma 5.2]) that N_s is a connected and simply-connected nilpotent Lie group, $N = N_s \cap G$, and $G/N \cong T$.

Theorem 3.4.1 ([15, Theorems 5.3 and 5.4]). *Let G be a connected and simply-connected solvable Lie group. Then G admits a unique semisimple splitting.*

We shall not give the whole proof of this theorem that can be found in [15]. But we shortly describe the construction of the semisimple splitting. In order to do this, we recall the Jordan decomposition of certain morphisms:

Let φ be an endomorphism of a finite-dimensional vector space over a field of characteristic zero. There is a unique *Jordan sum decomposition*

$$\varphi = \varphi_s + \varphi_n, \quad \varphi_s \circ \varphi_n = \varphi_n \circ \varphi_s,$$

with φ_s semisimple and φ_n nilpotent. They are called respectively the *semisimple part* and the *nilpotent part of φ* . If φ is an automorphism, it also has a unique *Jordan product decomposition*

$$\varphi = \varphi_s \circ \varphi_u, \quad \varphi_s \circ \varphi_u = \varphi_u \circ \varphi_s,$$

with φ_s semisimple and φ_u unipotent; φ_s is the same as in the sum decomposition and $\varphi_u = \text{id} + (\varphi_s^{-1} \circ \varphi_n)$. The latter is called the *unipotent part of φ* .

Note, if φ is a derivation resp. an automorphism of a Lie algebra, then the semisimple and the nilpotent resp. unipotent part of φ are also derivations resp. automorphisms of the Lie algebra.

Now, let G be a connected and simply-connected Lie group and $f: G \rightarrow G$ a Lie group automorphism.

Then $f_* := d_e f$ is a Lie algebra automorphism which has a Jordan product decomposition $f_* = (f_*)_s \circ (f_*)_u = (f_*)_u \circ (f_*)_s$. The *semisimple* and *unipotent part of f* are by definition the unique Lie group automorphisms $f_s, f_u: G \rightarrow G$ with $d_e f_s = (f_*)_s$ and $d_e f_u = (f_*)_u$.

Construction of the semisimple splitting. Let G be a connected and simply-connected solvable Lie group. Denote by N the nilradical of G .

By [15, Proposition 3.3], there exists a connected and simply-connected nilpotent Lie subgroup H of G such that $G = H \cdot N$. Fix such an H and consider the well-defined (!) action $\tilde{\omega}: H \rightarrow \text{A}(G)$ given by $\tilde{\omega}(a)(h \cdot n) := h \cdot (I_a|_N)_s(n)$, where $(I_a|_N)_s$ is the semisimple part of the automorphism of N which is obtained by conjugating every element of N by a .

Define $T := H/(N \cap H) \cong HN/N \cong G/N \cong \mathbb{R}^k$. Note that there is an action ϖ of T on G making the following diagram commutative:

$$\begin{array}{ccc}
 H & \xrightarrow{\tilde{\varpi}} & A(G) \\
 \pi \searrow & & \nearrow \varpi \\
 & T = H/(H \cap N) &
 \end{array}$$

Set $G_s := T \ltimes_{\varpi} G$.

One calculates that

$$N_s := \{\pi(h^{-1}) \cdot h \mid h \in H\} \cdot N = \{(\pi(h^{-1}), h \cdot n) \mid h \in H, n \in N\} \subset T \times G = G_s$$

is the nilradical of G_s . Furthermore, we have $T \cdot N_s = G_s$ and $T \cap N_s = \{e\}$. For $t \in T$, $h \in H$, $n \in N$ and every $h_t \in \pi^{-1}(\{t\})$ holds

$$\begin{aligned}
 \nu_s(t)(\pi(h^{-1}) \cdot (h \cdot n)) &:= t \cdot \pi(h^{-1}) \cdot (h \cdot n) \cdot t^{-1} \\
 &= \pi(h^{-1}) \cdot (h \cdot \tilde{\varpi}(h_t)(n)),
 \end{aligned}$$

i.e. $\nu_s(t)$ is a semisimple automorphism and $G_s = T \ltimes_{\nu_s} N_s$.

Remark. As usual, we denote the Lie algebras of the above Lie groups by the corresponding small German letters. In [21, Chapter III] can be found:

There exists a vector subspace V of $|\mathfrak{g}|$ with $|\mathfrak{g}| = V \oplus |\mathfrak{n}|$ as vector spaces and $\forall_{A, B \in V} \text{ad}(A)_s(B) = 0$, where $\text{ad}(A)_s$ denotes the semisimple part of $\text{ad}(A)$.

Let \mathfrak{v} be a copy of V , considered as abelian Lie algebra. Then the Lie algebra of the semisimple splitting for G is $\mathfrak{g}_s = \mathfrak{v} \ltimes_{\text{ad}(\dots)_s} \mathfrak{g}$, i.e.

$$\forall_{(A, X), (B, Y) \in \mathfrak{g}_s} [(A, X), (B, Y)] = (0, [X, Y] + \text{ad}(A)_s(Y) - \text{ad}(B)_s(X)),$$

with nilradical $\mathfrak{n}_s = \{(-X_V, X) \mid X \in \mathfrak{g}\}$, where X_V denotes the component of X in V .

Now we state the announced criterion for the existence of lattices in solvable Lie groups.

Theorem 3.4.2 ([2, p. 248]). *Let G be a connected and simply-connected solvable Lie group with nilradical N and semisimple splitting $G_s = T \ltimes_{\nu_s} N_s$, where N_s is the nilshadow of G .*

Then G/N is contained as a subgroup in $G_s/N = T \times (N_s/N)$ and the projections $\pi_1: G/N \rightarrow T$, $\pi_2: G/N \rightarrow N_s/N$ are isomorphisms of abelian Lie groups.

Moreover, G admits a lattice if and only if the following conditions are satisfied:

- (i) *There exists a basis $\mathfrak{X} := \{X_1, \dots, X_n, \dots, X_m\}$ with rational structure constants of the Lie algebra \mathfrak{n}_s of N_s such that $\{X_1, \dots, X_n\}$ is a basis of the Lie algebra \mathfrak{n} of N .*

We write $\mathfrak{n}_s(\mathbb{Q})$ for the rational Lie algebra $\langle \mathfrak{X} \rangle_{\mathbb{Q}}$ and $N_s(\mathbb{Q})$ for its image under the exponential map.

- (ii) *There exists a lattice subgroup Γ_T of T with $\Gamma_T \subset \pi_1 \circ \pi_2^{-1}(N_s(\mathbb{Q})/N)$ such that the natural action $\Gamma_T \rightarrow A(\mathfrak{n}_s(\mathbb{Q}))$ is described by integer matrices in an appropriate basis of $\mathfrak{n}_s(\mathbb{Q})$. \square*

3.5 Three-dimensional solvmanifolds

The only one- and two-dimensional solvmanifolds are tori. Therefore, we begin our studies of low-dimensional solvmanifolds in dimension three.

Proposition 3.5.1 ([3]). *Every 3-dimensional connected and simply-connected solvable non-nilpotent Lie group G that possesses a lattice Γ has a 2-dimensional nilradical. The Lie group can be written as $G = \mathbb{R} \ltimes_{\mu} \mathbb{R}^2$ and the lattice as $\Gamma = \mathbb{Z} \ltimes_{\mu} \mathbb{Z}^2$.*

Proof. This is a direct consequence of Proposition 3.2.5 and Corollary 3.3.5. \square

Theorem 3.5.2. *A three-dimensional solvmanifold G/Γ is non-formal if and only if $b_1(G/\Gamma) = 2$. In this case, G/Γ is diffeomorphic to a nilmanifold.*

Proof. By Theorem 3.1.7, it suffices to consider the case when G is solvable and non-nilpotent. The last proposition implies that there is a map $\nu: \mathbb{Z} \rightarrow \mathrm{SL}(2, \mathbb{Z})$ such that $\Gamma = \mathbb{Z} \ltimes_{\nu} \mathbb{Z}^2$.

If none of the roots of $\nu(1)$ equals 1, Proposition 3.3.7 implies $b_1 = 1$, so G/Γ is formal by Theorem 1.3.1.

Assume that $\nu(1)$ possesses the double root 1. Then Proposition 3.3.7 implies $b_1 = 3$ if $\nu(1)$ is diagonalisable and $b_1 = 2$ if $\nu(1)$ is not diagonalisable.

Case A: $\nu(1)$ is diagonalisable

Recall that a solvmanifold is uniquely determined by its fundamental group. Therefore, we can assume $G = \mathbb{R} \ltimes_{\mu} \mathbb{R}^2$ and $\Gamma = \mathbb{Z} \ltimes_{\mu} \langle v_1, v_2 \rangle_{\mathbb{Z}}$ with linearly independent $v_1, v_2 \in \mathbb{R}^2$ and $\mu(t) \equiv \mathrm{id}$. In this case, G/Γ is a torus which is formal.

Case B: $\nu(1)$ is not diagonalisable

In this case, we can assume $G = \mathbb{R} \ltimes_{\mu} \mathbb{R}^2$ as well as $\Gamma = \mathbb{Z} \ltimes_{\mu} \langle v_1, v_2 \rangle_{\mathbb{Z}}$ with linearly independent $v_1, v_2 \in \mathbb{R}^2$ and

$$\mu(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g} = \langle T, X, Y \mid [T, Y] = X \rangle$ of G is nilpotent, so G/Γ is a nilmanifold with $b_1 = 2$. Therefore, it cannot be a torus and is not formal by Theorem 3.1.7. \square

In [3, Chapter III §3] the three-dimensional solvmanifolds which have no nilmanifold structure are examined. This, together with the last theorem, yields a “cohomological” classification of three-dimensional solvmanifolds.

Theorem 3.5.3. *Every 3-dimensional solvmanifold G/Γ is contained in Table 3.1 on page 39. In particular, G/Γ is non-formal if and only if it is a non-toral*

Table 3.1: 3-dimensional solvmanifolds

	$b_1(G/\Gamma)$	G/Γ formal	Nilmf. ³	c.s. ⁴
a)	3	yes	Torus	yes
b)	2	no	yes	yes
c)	1	yes	no	yes
d)	1	yes	no	no

nilmanifold. \square

Example. The torus $\mathbb{R}^3/\mathbb{Z}^3$ is a solvmanifold with $b_1 = 3$, and examples of 3-dimensional solvmanifolds with $b_1 = 2$ will be given in the next theorem.

For $i \in \{1, 2\}$ consider the Lie groups $G_i = \mathbb{R} \ltimes_{\mu_i} \mathbb{R}^2$, where μ_i is given by $\mu_1(t)(x, y) = (e^t x, e^{-t} y)$, $\mu_2(t)(x, y) = (\cos(t) x + \sin(t) y, -\sin(t) x + \cos(t) y)$.

G_1 is completely solvable and possesses the lattice

$$\Gamma_1 := t_1 \mathbb{Z} \ltimes_{\mu_1} \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{18+8\sqrt{5}}{7+3\sqrt{5}} \\ \frac{2}{3+\sqrt{5}} \end{pmatrix} \right\rangle_{\mathbb{Z}},$$

where $t_1 = \ln(\frac{3+\sqrt{5}}{2})$. Note that the following equation implies that Γ_1 really is a lattice

$$\begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{18+8\sqrt{5}}{7+3\sqrt{5}} \\ 1 & \frac{2}{3+\sqrt{5}} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & \frac{18+8\sqrt{5}}{7+3\sqrt{5}} \\ 1 & \frac{2}{3+\sqrt{5}} \end{pmatrix}^{-1}. \quad (3.4)$$

G_2 is not completely solvable and contains the lattice

$$\Gamma_2 = \pi \mathbb{Z} \ltimes_{\mu_2} \mathbb{Z}^2.$$

A short computation yields that the abelianisations of Γ_i have rank one, i.e. we have constructed examples of type c) and d) in Table 3.1.

³possesses the structure of a solvmanifold as quotient of a nilpotent Lie group

⁴possesses the structure of a solvmanifold as quotient of a completely solvable Lie group

Theorem 3.5.4. *Every lattice in the unique 3-dimensional connected and simply-connected non-abelian nilpotent Lie group*

$$U_3(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

is isomorphic to $\Gamma_{3,n} := \Gamma_{3,n}(\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & x & \frac{z}{n} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$ with $n \in \mathbb{N}_+$.

Therefore, any three-dimensional nilmanifold with $b_1 = 2$ is of the form $U_3(\mathbb{R})/\Gamma_{3,n}(\mathbb{Z})$.

$\Gamma_{3,n}(\mathbb{Z})$ is presented by $\langle e_1, e_2, e_3 \mid [e_1, e_2] = e_3^n \text{ and } e_3 \text{ central} \rangle$.

Proof. $U_3(\mathbb{R})$ is the only connected and simply-connected non-abelian nilpotent Lie group of dimension three. By [3, Chapter III § 7], each lattice in it is isomorphic to $\Gamma_{3,n}$. The other assertions follow trivially. \square

Sometimes, we shall write (x, y, z) for the corresponding matrix in $U_3(\mathbb{R})$.

For later applications, we are going to determine the Lie group automorphisms and the one-parameter groups of $U_3(\mathbb{R})$. In order to do this, we start with the following proposition. Note that $Z(G)$ denotes the center of a group G .

Proposition 3.5.5.

- (i) $[U_3(\mathbb{R}), U_3(\mathbb{R})] = Z(U_3(\mathbb{R})) = \{(0, 0, z) \mid z \in \mathbb{R}\}$, $U_3(\mathbb{R})/Z(U_3(\mathbb{R})) \cong \mathbb{R}^2$
- (ii) Every Lie group homomorphism $f: U_3(\mathbb{R}) \rightarrow U_3(\mathbb{R})$ induces natural Lie group homomorphisms

$$f_Z: Z(U_3(\mathbb{R})) \longrightarrow Z(U_3(\mathbb{R}))$$

and

$$\begin{aligned} \bar{f}: U_3(\mathbb{R})/Z(U_3(\mathbb{R})) &\longrightarrow U_3(\mathbb{R})/Z(U_3(\mathbb{R})). \\ [(x, y, 0)] = [(x, y, z)] &\longmapsto [f((x, y, z))] = [(f_1(x, y, 0), f_2(x, y, 0), 0)] \end{aligned}$$

\bar{f} uniquely determines f_Z , and \bar{f} is an automorphism if and only if f is such.

- (iii) Let $\gamma_1 = (a_1, b_1, \frac{c_1}{n}), \gamma_2 = (a_2, b_2, \frac{c_2}{n}) \in \Gamma_{3,n}$. Then there is a unique homomorphism $g: \Gamma_{3,n} \rightarrow \Gamma_{3,n}$ such that $g((1, 0, 0)) = \gamma_1$ and $g((0, 1, 0)) = \gamma_2$. Moreover, $g((0, 0, \frac{1}{n})) = (0, 0, \frac{1}{n}(a_1 b_2 - a_2 b_1))$.

One has $\Gamma_{3,n}/Z(\Gamma_{3,n}) \cong \mathbb{Z}^2$, and g is an isomorphism if and only if

$$\bar{g}: \Gamma_{3,n}/Z(\Gamma_{3,n}) \longrightarrow \Gamma_{3,n}/Z(\Gamma_{3,n})$$

is an isomorphism, i.e. $a_1 b_2 - a_2 b_1 = \pm 1$.

Proof. (i) is trivial.

ad (ii): Let $f: U_3(\mathbb{R}) \rightarrow U_3(\mathbb{R})$ be a Lie group homomorphism. Then

$$f((0, 0, z)) = [f((z, 0, 0)), f((0, 1, 0))] \in Z(U_3(\mathbb{R})), \quad (3.5)$$

i.e. $f(Z(U_3(\mathbb{R}))) \subset Z(U_3(\mathbb{R}))$. Moreover, one has for $(a, b, c) := f((x, y, 0))$

$$(a, b, 0)^{-1} \cdot (a, b, c) = (-a, -b, -ab) \cdot (a, b, c) = (0, 0, -2ab + c) \in Z(U_3(\mathbb{R})),$$

and therefore $[(a, b, 0)] = \overline{f}([(x, y, 0)])$. Now, (3.5) implies that f_Z is uniquely determined by \overline{f} .

Assume, f is an isomorphism. Then (3.5) also holds for f^{-1} and we get $f(Z(U_3(\mathbb{R}))) = Z(U_3(\mathbb{R}))$, i.e. f_Z is an isomorphism of the additive group \mathbb{R} . Since f is continuous, there exists $m \in \mathbb{R} \setminus \{0\}$ such that $f_Z((0, 0, z)) = (0, 0, mz)$. Denote by $(f_{ij})_{1 \leq i, j \leq 2}$ the matrix of $\overline{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with respect to the basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ of the vector space \mathbb{R}^2 . One calculates

$$\begin{aligned} (0, 0, \det(f_{ij})) &= [(f_{11}, f_{21}, 0), (f_{12}, f_{22}, 0)] = [f((1, 0, 0)), f((0, 1, 0))] \\ &\stackrel{(3.5)}{=} (0, 0, m), \end{aligned}$$

so \overline{f} is an automorphism, since $m \neq 0$.

Conversely, if \overline{f} is an automorphism, then the homomorphism f_Z is given by $f_Z((0, 0, z)) = (0, 0, \det(\overline{f})z)$ which is even an automorphism. Therefore, the 5-Lemma implies that f is an automorphism.

ad (iii): Let γ_1, γ_2 be as in (iii). Then $[\gamma_1, \gamma_2] = (0, 0, \frac{1}{n}(a_1b_2 - a_2b_1))^n$ and this implies the existence of the (unique) homomorphism g with the mentioned properties.

If g is an isomorphism, then $g(Z(\Gamma_{3,n})) = Z(\Gamma_{3,n}) = \{(0, 0, \frac{z}{n}) \mid z \in \mathbb{Z}\}$, and therefore $|a_1b_2 - a_2b_1| = 1$. Since the matrix of \overline{g} has determinant $a_1b_2 - a_2b_1$, \overline{f} is an isomorphism.

Again, the converse is trivial. \square

Theorem 3.5.6. *As a set, the group of Lie group automorphisms $A(U_3(\mathbb{R}))$ equals $GL(2, \mathbb{R}) \times \mathbb{R}^2$, the group law is given by*

$$(A, a) \circ (B, b) = (AB, \det(B)B^{-1}a + \det(A)b), \quad (3.6)$$

and for $f = (A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}) \in A(U_3(\mathbb{R}))$ and $(x, y, z) \in U_3(\mathbb{R})$ we have

$$f((x, y, z)) = \left(\alpha x + \beta y, \gamma x + \delta y, \det(A)z + \beta\gamma xy + \frac{\alpha\gamma}{2}x^2 + \frac{\beta\delta}{2}y^2 + uy - vx \right). \quad (3.7)$$

Proof. Let $f \in A(U_3(\mathbb{R}))$ and $(x, y, z) \in U_3(\mathbb{R})$ be given. We have to show that there is $\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right) \in \text{GL}(2, \mathbb{R}) \times \mathbb{R}^2$ such that $f((x, y, z))$ satisfies (3.7). Then a short computation yields (3.6).

Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}(2, \mathbb{R})$ be the matrix of \bar{f} with respect to the canonical basis of \mathbb{R}^2 . We showed in the last proof $f((0, 0, z)) = (0, 0 \det(\bar{f})z)$.

There exist smooth functions $f_1, f_2: \mathbb{R} \rightarrow \mathbb{R}$ with

$$\begin{aligned} f((x, 0, 0)) &= (\alpha x, \gamma x, f_1(x)), \\ f((0, y, 0)) &= (\beta y, \delta y, f_2(y)). \end{aligned}$$

We set $u := f_2'(0)$ and $v := -f_1'(0)$. The homomorphism property of f implies

$$\begin{aligned} \frac{1}{h}(f_1(x+h) - f_1(x)) &= \frac{f_1(h) - f_1(0)}{h} + \alpha\gamma x, \\ \frac{1}{h}(f_2(y+h) - f_2(y)) &= \frac{f_2(h) - f_2(0)}{h} + \beta\delta y, \end{aligned}$$

and this yields

$$\begin{aligned} f_1(x) &= -vx + \frac{\alpha\gamma}{2}x^2, \\ f_2(y) &= uy + \frac{\beta\delta}{2}y^2. \end{aligned}$$

Using $(x, y, z) = (0, y, 0)(x, 0, 0)(0, 0, z)$, one computes (3.7). \square

Corollary 3.5.7. $f = (A, \begin{pmatrix} u \\ v \end{pmatrix}) \in A(U_3(\mathbb{R}))$ with $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ lies on a one-parameter group of $A(U_3(\mathbb{R}))$ if and only if A lies on a one-parameter group of $\text{GL}(2, \mathbb{R})$.

If $\nu_t = \begin{pmatrix} \alpha_t & \beta_t \\ \gamma_t & \delta_t \end{pmatrix}$ denotes a one-parameter group with $\nu_1 = A$, then the map $\mu_t: \mathbb{R} \rightarrow A(U_3(\mathbb{R}))$ defined by

$$\begin{aligned} \mu_t((x, y, z)) &= (\alpha_t x + \beta_t y, \gamma_t x + \delta_t y, \\ &\quad \underbrace{(\alpha_t \delta_t - \beta_t \gamma_t)}_{=1} z + \beta_t \gamma_t x y + \frac{\alpha_t \gamma_t}{2} x^2 + \frac{\beta_t \delta_t}{2} y^2 + t u y - t v x) \end{aligned}$$

is a one-parameter group with $\mu_1 = f$.

Proof. The only claim that is not obvious is the fact that μ_t defines a one-parameter group. Using $\nu_{t+s} = \nu_t \circ \nu_s$, this can be seen by a short calculation. \square

3.6 Four-dimensional solvmanifolds

As we have done in the three-dimensional case, we are going to give a “cohomological” classification of four-dimensional solvmanifolds. We shall consider all isomorphism classes of lattices that can arise in a four-dimensional connected and simply-connected solvable Lie group. The next proposition describes such lattices in the case of a non-nilpotent group.

Proposition 3.6.1. *Every 4-dimensional connected and simply-connected solvable non-nilpotent Lie group G that possesses a lattice Γ has a 3-dimensional nilradical N which is either \mathbb{R}^3 or $U_3(\mathbb{R})$. Therefore, G/Γ fibers over S^1 (this is the Mostow bundle) and the Lie group can be written as $G = \mathbb{R} \ltimes_{\mu} N$. If N is abelian, a basis transformation yields $\Gamma = \mathbb{Z} \ltimes_{\mu|_{\mathbb{Z}^3}} \mathbb{Z}^3$. Otherwise, Γ is isomorphic to $\mathbb{Z} \ltimes_{\nu} \Gamma_{3,n}$, where $\nu: \mathbb{Z} \rightarrow \text{Aut}(\Gamma_{3,n})$ is a group homomorphism with*

$$\nu(1)(x, y, \frac{z}{n}) = \left(a_1x + a_2y, \quad b_1x + b_2y, \quad a_2b_1xy + a_1b_1\frac{x(x-1)}{2} + a_2b_2\frac{y(y-1)}{2} + \frac{1}{n}(c_1x + c_2y + (a_1b_2 - a_2b_1)z) \right),$$

where $c_1, c_2, \in \mathbb{Z}$, and $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$ is the matrix of $\overline{\nu(1)}$ with respect to the canonical basis of the \mathbb{Z} -module $\mathbb{Z}^2 = \Gamma_{3,n}/Z(\Gamma_{3,n})$. Moreover, $\overline{\nu(1)}$ lies on a one-parameter group $\mathbb{R} \rightarrow \text{A}(U_3(\mathbb{R})/Z(U_3(\mathbb{R}))) = \text{GL}(2, \mathbb{R})$, i.e. $\overline{\nu(1)} \in \text{SL}(2, \mathbb{R})$.

Proof. From [63, Theorem 3.1.10] follows $\dim N = 3$ and $G = \mathbb{R} \ltimes_{\mu} N$. If N is abelian, Corollary 3.3.5 implies that we can assume $\Gamma = \mathbb{Z} \ltimes_{\mu|_{\mathbb{Z}^3}} \mathbb{Z}^3$.

Assume now that N is not abelian, i.e. $N = U_3(\mathbb{R})$. $\Gamma_N = \Gamma \cap N$ is a lattice in N and by Theorem 3.5.4, we have $\Gamma_N = \Gamma_{3,n}$. By Corollary 3.3.5, there is a homomorphism $\nu: \mathbb{Z} \rightarrow \text{Aut}(\Gamma_{3,n})$ with $\Gamma \cong \mathbb{Z} \ltimes_{\nu} \Gamma_{3,n}$. Proposition 3.5.5(iii) implies that $\nu(1)$ is determined by $(a_1, b_1, \frac{c_1}{n}) := \nu(1)((1, 0, 0))$ and $(a_2, b_2, \frac{c_2}{n}) := \nu(1)((0, 1, 0)) \in \Gamma_{3,n}$. Since $(x, y, \frac{z}{n}) = (0, 1, 0)^y(1, 0, 0)^x(0, 0, \frac{1}{n})^z$, a short computation yields the claimed formula for $\nu(1)((x, y, \frac{z}{n}))$.

Further, Corollary 3.5.7 implies that $\overline{\nu(1)}$ lies on a one-parameter group. \square

Theorem 3.6.2. *Every 4-dimensional solvmanifold G/Γ is contained in Table 3.2. In particular, G/Γ is non-formal if and only if it is a non-toral nilmanifold.*

Proof. Apart from the column on formality the theorem follows from works of Geiges [31] and Hasegawa [40]. (Attention: In [40] a more general notion of solvmanifold is used!)

A decomposable four-dimensional connected and simply-connected nilpotent Lie group is abelian or has a two-dimensional center. The only connected and simply-connected indecomposable nilpotent Lie group of dimension four has a two-dimensional commutator. By Propositions 3.1.4 and 3.1.3, the corresponding nilmanifolds have the structure of orientable T^2 -bundles over T^2 . (The orientability follows from the total spaces’ orientability.)

Table 3.2: 4-dimensional solvmanifolds

	$b_1(G/\Gamma)$	G/Γ formal	symplectic	complex	Kähler	Nilmf. ⁵	c.s. ⁶
a)	4	yes	yes	Torus	yes	Torus	yes
b)	3	no	yes	PKS ⁷	no	yes	yes
c)	2	yes	yes	no	no	no	yes
d)	2	yes	yes	HS ⁸	yes	no	no
e)	2	no	yes	no	no	yes	yes
f)	1	yes	no	no	no	no	yes
g)	1	yes	no	IS ⁰ ⁹	no	no	no
h)	1	yes	no	IS ⁺ ¹⁰	no	no	yes
i)	1	yes	no	SKS ¹¹	no	no	no

From a result of Geiges [31, Theorems 1 and 3] follows that they are contained in Table 3.2. (Recall that a nilmanifold is formal if and only if it is a torus.) In particular, every four-dimensional nilmanifold is symplectic.

Now, we regard a lattice $\Gamma = \mathbb{Z} \ltimes_{\nu} \Gamma_N$, $\Gamma_N \in \{\mathbb{Z}^3, \Gamma_{3,n}(\mathbb{Z})\}$, in a Lie group $G = \mathbb{R} \ltimes_{\mu} N$ as in the last proposition.

We expand Hasegawa's argumentation in [40] by the aspect of formality and consider the "roots" of $\nu(1)$. Recall, Corollary 3.2.2 implies that a solvmanifold is determined by its fundamental group. Below, we shall use this fact several times.

Case A.: $\Gamma_N = \mathbb{Z}^3$

By Proposition 3.6.1, ν extends to a one-parameter group $\mathbb{R} \rightarrow \mathrm{SL}(3, \mathbb{R})$. Denote by $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ the roots of $\nu(1) \in \mathrm{SL}(3, \mathbb{Z})$, i.e. $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1$. We get from Theorem 3.3.3 and Lemma B.4 that the following subcases can occur:

A.1.) $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+$

A.1.1.) $\exists_{i_0} \lambda_{i_0} = 1$ (w.l.o.g. $\lambda_1 = 1$)

A.1.1.1.) $\lambda_2 = \lambda_3 = 1$

A.1.1.2.) $\lambda_2 = \lambda_3^{-1} \in \mathbb{R} \setminus \{1\}$

A.1.2.) $\forall_i \lambda_i \neq 1$

A.1.2.1.) $\nu(1)$ is diagonalisable

A.1.2.2.) $\nu(1)$ is not diagonalisable

⁵possesses the structure of a solvmanifold as quotient of a nilpotent Lie group

⁶possesses the structure of a solvmanifold as quotient of a completely solvable Lie group

⁷Primary Kodaira Surface

⁸Hyperelliptic Surface

⁹Inoue Surface of Type \mathbf{S}^0

¹⁰Inoue Surface of Type \mathbf{S}^+

¹¹Secondary Kodaira Surface

A.2.) $\lambda_1 = 1, \lambda_2 = \lambda_3 = -1$ and $\nu(1)$ is diagonalisable

A.3.) $\exists_{i_0} \lambda_{i_0} \in \mathbb{C} \setminus \mathbb{R}$ (w.l.o.g. $\lambda_2 = \overline{\lambda_3} \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda_1 \in \mathbb{R}_+$)

A.3.1.) $\lambda_1 = 1$

A.3.2.) $\lambda_1 \neq 1$

Case A.1.1.1.: $\lambda_1 = \lambda_2 = \lambda_3 = 1$

If $\nu(1)$ is diagonalisable, then G/Γ is clearly a torus. This is case a). If $\nu(1)$ is not diagonalisable we can assume $G = \mathbb{R} \rtimes_{\mu} \mathbb{R}^3$ and $\Gamma = \mathbb{Z} \rtimes_{\mu} \langle v_1, v_2, v_3 \rangle_{\mathbb{Z}}$, where $\mu(t)$ is one of the following one-parameter groups

$$\begin{aligned} \exp\left(t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \\ \exp\left(t \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) &= \begin{pmatrix} 1 & t & \frac{1}{2}(t^2 - t) \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

In both cases G/Γ is a 4-dimensional nilmanifold and therefore symplectic. In the first case, we have a primary Kodaira surface with $b_1 = 3$, see [40, Section 2.2.3)]; in the second case the nilmanifold has $b_1 = 2$ and no complex structure, see [41, Example 2]. Being non-toral nilmanifolds, both are not formal and we get the cases b) and e).

Cases A.1.1.2. and A.1.2.1.: The λ_i are positive and pairwise different or two of them are equal but $\nu(1)$ is diagonalisable. (The latter cannot happen by Lemma B.4.)

We can assume $G = \mathbb{R} \rtimes_{\mu} \mathbb{R}^3$ and $\Gamma = \mathbb{Z} \rtimes_{\mu} \langle v_1, v_2, v_3 \rangle_{\mathbb{Z}}$ with linearly independent $v_1, v_2, v_3 \in \mathbb{R}^3$, where $\mu(t) = \begin{pmatrix} \exp(t \ln(\lambda_1)) & 0 & 0 \\ 0 & \exp(t \ln(\lambda_2)) & 0 \\ 0 & 0 & \exp(t \ln(\lambda_3)) \end{pmatrix}$. By [41, Example 2], the solvmanifold G/Γ does not admit a complex structure.

One computes the Lie algebra of G as

$$\mathfrak{g} = \langle T, X, Y, Z \mid [T, X] = \ln(\lambda_1)X, [T, Y] = \ln(\lambda_2)Y, [T, Z] = \ln(\lambda_3)Z \rangle$$

which is completely solvable and non-nilpotent. Therefore, the minimal model of the Chevalley-Eilenberg complex is the minimal model of G/Γ .

If none of the roots λ_i is one, we see by Proposition 3.3.7 that $b_1(G/\Gamma) = 1$. Since G/Γ is parallelisable, this implies $b_2(G/\Gamma) = 0$, so the space cannot be symplectic. Further it is formal by Theorem 1.3.1. This is case f) in Table 3.2.

If one of the roots is one (w.l.o.g. $\lambda_1 = 1$), we have $b_1(G/\Gamma) = 2$ and the Chevalley-Eilenberg complex is

$$\left(\bigwedge (\tau, \alpha, \beta, \gamma), d\tau = d\alpha = 0, d\beta = -\ln(\lambda_2) \tau \wedge \beta, d\gamma = -\ln(\lambda_3) \tau \wedge \gamma \right).$$

$\tau \wedge \alpha + \alpha \wedge \beta + \alpha \wedge \gamma - \beta \wedge \gamma$ defines a symplectic form on G/Γ . Further, one computes the first stage of the minimal model of the Chevalley-Eilenberg complex as

$$\mathcal{M}^{\leq 1} = \bigwedge (x_1, x_2), \quad dx_i = 0.$$

Therefore, G/Γ is 1-formal and by Theorem 1.1.6 formal. This is case c) in Table 3.2.

Case A.1.2.2.: $\lambda_i \in \mathbb{R}_+ \setminus \{1\}$ and $\nu(1)$ is not diagonalisable

In this case two roots must be equal (w.l.o.g. $\lambda_2 = \lambda_3$) and the third is different from the others, i.e. $\lambda_1 = \frac{1}{\lambda_2} \neq \lambda_2$. Since the characteristic polynomial of $\nu(1)$ has integer coefficients, Lemma B.4 implies $\lambda_2 = \pm 1$ and this is a contradiction.

Cases A.2. and A.3.1.: $\lambda_1 = 1, \lambda_2 = \overline{\lambda_3} = \exp(i\varphi) \in \mathbb{C} \setminus \mathbb{R}, \varphi \in]0, 2\pi[$

We can assume $G = \mathbb{R} \rtimes_{\mu} \mathbb{R}^3$ and $\Gamma = \mathbb{Z} \rtimes_{\mu} \langle v_1, v_2, v_3 \rangle_{\mathbb{Z}}$ with linearly independent

$v_1, v_2, v_3 \in \mathbb{R}^3$, where $\mu(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t\varphi) & -\sin(t\varphi) \\ 0 & \sin(t\varphi) & \cos(t\varphi) \end{pmatrix}$. Thus G/Γ is a hyper-

elliptic surface (see [40, Section 3.3.]) which is Kählerian and has $b_1 = 2$. The Lie algebra of G is not completely solvable and we are in case d).

Case A.3.2.: $\lambda_1 \neq 1, \lambda_2 = \overline{\lambda_3} = |\lambda_2| \exp(i\varphi) \in \mathbb{C} \setminus \mathbb{R}, \varphi \in]0, 2\pi[\setminus \{\pi\}$

We can assume $G = \mathbb{R} \rtimes_{\mu} \mathbb{R}^3$ and $\Gamma = \mathbb{Z} \rtimes_{\mu} \langle v_1, v_2, v_3 \rangle_{\mathbb{Z}}$ with linearly independent

$v_1, v_2, v_3 \in \mathbb{R}^3$, where $\mu(t) = \begin{pmatrix} \lambda_1^t & 0 & 0 \\ 0 & |\lambda_2|^t \cos(t\varphi) & -|\lambda_2|^t \sin(t\varphi) \\ 0 & |\lambda_2|^t \sin(t\varphi) & |\lambda_2|^t \cos(t\varphi) \end{pmatrix}$. Thus G/Γ is

a Inoue surface of type S^0 (see [40, Section 3.6.]), which is not symplectic and has $b_1 = 1$ (by Proposition 3.3.7, since 1 is no root of $\mu(1)$). By Theorem 1.3.1, G/Γ is formal. The Lie algebra of G is not completely solvable and this yields case g) of Table 3.2.

Case B.: $\Gamma_N = \Gamma_{3,n}(\mathbb{Z})$

In this case we have a homomorphism $\nu: \mathbb{Z} \rightarrow \text{Aut}(\Gamma_{3,n}(\mathbb{Z}))$. We shall write N for $U_3(\mathbb{R})$ and Γ_N for $\Gamma_{3,n}(\mathbb{Z})$. The automorphism $\nu(1)$ induces an automorphism $\overline{\nu}(1)$ of $\Gamma_N/Z(\Gamma_N) = \mathbb{Z}^2$ which lies by Proposition 3.6.1 on a one-parameter group $\mathbb{R} \rightarrow \text{A}(U_3(\mathbb{R})/\Gamma_N) = \text{GL}(2, \mathbb{R})$. Denote the roots of $\overline{\nu}(1) \in \text{GL}(2, \mathbb{Z})$ by $\tilde{\lambda}_1, \tilde{\lambda}_2$. Theorem 3.3.3 shows that the following cases are possible:

B.1.) $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \mathbb{R}_+$

B.1.1.) $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$

B.1.2.) $\tilde{\lambda}_1 = \tilde{\lambda}_2^{-1} \neq 1$

B.2.) $\tilde{\lambda}_1 = \tilde{\lambda}_2 = -1$ and $\overline{\nu}(1)$ is diagonalisable

B.3.) $\tilde{\lambda}_1 = \overline{\tilde{\lambda}_2} \in \mathbb{C} \setminus \mathbb{R}$

$\nu(1)$ also induces an automorphism $\nu(1)_Z$ of the center $Z(\Gamma_N)$ of Γ_N which equals $\det(\bar{\nu}(1)) \cdot \text{id} = \text{id}$ by Proposition 3.5.5(iii).

Case B.1.1.: $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$

By [30, Lemma 1], we can assume $\bar{\nu}(1) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ with $k \in \mathbb{N}$. Then Proposition 3.6.1 yields $\nu(1)((x, y, \frac{z}{n})) = (x + ky, y, k\frac{y(y-1)}{2} + \frac{c_1x + c_2y + z}{n})$ and this implies

$$\begin{aligned} \Gamma &= \mathbb{Z} \rtimes_{\nu} \Gamma_{3,n} \\ &= \langle e_0, \dots, e_3 \mid [e_0, e_1] = e_3^{c_1}, [e_2^{-1}, e_0] = e_1^k e_3^{c_2}, [e_0, e_3] = 1, [e_1, e_2] = e_3^n \rangle. \end{aligned}$$

This is a discrete torsion-free nilpotent group, which can be embedded as a lattice in a connected and simply-connected nilpotent Lie group by [66, Theorem 2.18]. Since a solvmanifold is uniquely determined by its fundamental group, G/Γ is diffeomorphic to a nilmanifold.

As at the beginning of the proof, we conclude that G/Γ is the total space of a T^2 -bundle over T^2 and occurs in our list. The quotient G/Γ is of type b) if $k = 0$ and of type e) if $k \neq 0$.

Case B.1.2.: $\tilde{\lambda}_1 = \tilde{\lambda}_2^{-1} \in \mathbb{R}_+ \setminus \{1\}$

We have $\bar{\nu}(1) = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$, and Proposition 3.6.1 implies

$$\nu(1)(x, y, \frac{z}{n}) = \left(a_1x + a_2y, \quad b_1x + b_2y, \quad a_2b_1xy + a_1b_1\frac{x(x-1)}{2} + a_2b_2\frac{y(y-1)}{2} + \frac{1}{n}(c_1x + c_2y + (a_1b_2 - a_2b_1)z) \right)$$

for certain $c_1, c_2 \in \mathbb{Z}$.

Choose eigenvectors $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$ with respect to the eigenvalues $\tilde{\lambda}_1$ resp. $\tilde{\lambda}_2$ of ${}^t\bar{\nu}(1)$ (where t denotes the transpose). There exist $u_1, u_2, u_3 \in \mathbb{R}$ such that for $\gamma_i := (v_i, w_i, u_i)$, $i \in \{1, 2\}$, and $\gamma_3 := (0, 0, u_3) \in U_3(\mathbb{R})$ we have

$$\begin{aligned} [\gamma_1, \gamma_2] &= \gamma_3^n, \\ \tilde{\mu}(1)(\gamma_1) &= \gamma_1^{a_1} \gamma_2^{b_1} \gamma_3^{c_1}, \quad \tilde{\mu}(1)(\gamma_2) = \gamma_1^{a_2} \gamma_2^{b_2} \gamma_3^{c_2}, \end{aligned}$$

where $\tilde{\mu}(t)((x, y, z)) = (\exp(t \ln(\tilde{\lambda}_1))x, \exp(t \ln(\tilde{\lambda}_2))y, z)$.

Then G/Γ is diffeomorphic to the solvmanifold $\tilde{G}/\tilde{\Gamma}$, where $\tilde{G} = \mathbb{R} \rtimes_{\tilde{\mu}} U_3(\mathbb{R})$ and $\tilde{\Gamma} = \mathbb{Z} \rtimes_{\tilde{\mu}} \langle \gamma_1, \gamma_2, \gamma_3 \rangle$, i.e. G/Γ is a Inoue surface of type S^+ , see [40, Section 3.7]. The Lie algebra of \tilde{G} ,

$$\tilde{\mathfrak{g}} = \langle T, X, Y, Z \mid [T, X] = X, [T, Y] = -Y, [X, Y] = Z \rangle,$$

is completely solvable and not nilpotent. Further, the knowledge of $\tilde{\mathfrak{g}}$ implies $b_1(G/\Gamma) = 1$. By Theorem 1.3.1, G/Γ is formal. Therefore, this is a solvmanifold of type h) in Table 3.2.

Case B.2.: $\tilde{\lambda}_1 = \tilde{\lambda}_2 = -1$ and $\bar{\nu}(1)$ is diagonalisable

[30, Lemma 1] implies that we can assume $\bar{\nu}(1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. So Proposition 3.6.1 implies $\nu(1)((x, y, \frac{z}{n})) = (-x, -y, \frac{c_1x+c_2y+z}{n})$ for certain integers $c_1, c_2 \in \mathbb{Z}$. Moreover, G/Γ is diffeomorphic to $\tilde{G}/\tilde{\Gamma}$, where $\tilde{G} := \mathbb{R} \rtimes_{\tilde{\mu}} U_3(\mathbb{R})$,

$$\tilde{\mu}(t)((x, y, z)) = (\cos(t\pi)x - \sin(t\pi)y, \sin(t\pi)x + \cos(t\pi)y, z + h_t(x, y)),$$

$h_t(x, y) = \frac{1}{2} \sin(t\pi) (\cos(t\pi)(x^2 - y^2) - 2\sin(t\pi)xy)$ and $\tilde{\Gamma} = \mathbb{Z} \rtimes_{\tilde{\mu}} \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ such that $[\gamma_1, \gamma_2] = \gamma_3^n$, $\tilde{\mu}(1)(\gamma_1) = \gamma_1^{-1}\gamma_3^{c_1}$ and $\tilde{\mu}(1)(\gamma_2) = \gamma_2^{-1}\gamma_3^{c_2}$. (Using the addition theorems for sin and cos, one calculates that $\tilde{\mu}$ is a one-parameter group in $A(U_3(\mathbb{R}))$.) By [40, Section 3.5], $\tilde{G}/\tilde{\Gamma}$ is a secondary Kodaira surface.

Obviously, the Lie algebra of \tilde{G} is not completely solvable and we cannot use its Chevalley-Eilenberg complex for computing $b_1(G/\Gamma)$. But since

$$\Gamma = \langle e_0, \dots, e_3 \mid e_0e_1e_0^{-1} = e_1^{-1}e_3^{c_1}, e_0e_2e_0^{-1} = e_2^{-1}e_3^{c_2}, [e_0, e_3] = 1, [e_1, e_2] = e_3^n \rangle,$$

we see $b_1(G/\Gamma) = \mathrm{rank} \Gamma_{ab} = 1$ and G/Γ belongs to the last row in Table 3.2.

Case B.3.: $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \exp(i\varphi) \in \mathbb{C} \setminus \mathbb{R}, \varphi \in]0, 2\pi[\setminus \{\pi\}$

This case is similar to the last one. We have $|\mathrm{tr} \bar{\nu}(1)| \leq |\tilde{\lambda}_1| + |\tilde{\lambda}_2| = 2$ and [30, Lemma 1] implies that we can assume $\nu(1)$ to be $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. In each case, one computes $b_1(G/\Gamma) = \mathrm{rank} \Gamma_{ab} = 1$, as above. Moreover, one embeds a lattice $\tilde{\Gamma}$ isomorphic to Γ in the Lie group $\tilde{G} := \mathbb{R} \rtimes_{\tilde{\mu}} U_3(\mathbb{R})$, where

$$\tilde{\mu}(t)((x, y, z)) = (\cos(t\varphi)x - \sin(t\varphi)y, \sin(t\varphi)x + \cos(t\varphi)y, z + h_t(x, y)),$$

$h_t(x, y) = \frac{1}{2} \sin(t\varphi) (\cos(t\varphi)(x^2 - y^2) - 2\sin(t\varphi)xy)$. Again, $\tilde{G}/\tilde{\Gamma}$ is a secondary Kodaira surface and G/Γ is an example for case i). For more details see [40, Section 3.5]. \square

Below, we give examples for each of the nine types of four-dimensional solvmanifolds. The Lie algebras of the connected and simply-connected four-dimensional solvable Lie groups that admit lattices are listed in Table A.1 in Appendix A.

Example. The following manifolds belong to the corresponding row in Table 3.2.

a) $\mathbb{R}^4/\mathbb{Z}^4$

b) $(\mathbb{R} \rtimes_{\mu_b} \mathbb{R}^3)/(\mathbb{Z} \rtimes_{\mu_b} \mathbb{Z}^3)$, $\mu_b(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

c) $(\mathbb{R} \rtimes_{\mu_c} \mathbb{R}^3)/\Gamma_c$ with

$$\Gamma_c = \mathbb{Z} \rtimes_{\mu_c} \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{18+8\sqrt{5}}{7+3\sqrt{5}} \\ \frac{2}{3+\sqrt{5}} \end{pmatrix} \right\rangle_{\mathbb{Z}},$$

$t_1 = \ln(\frac{3+\sqrt{5}}{2})$ and $\mu_c(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{tt_1} & 0 \\ 0 & 0 & e^{-tt_1} \end{pmatrix}$; the proof that this is really a solvmanifold is analogous to that in the example on page 39.

d) $(\mathbb{R} \rtimes_{\mu_d} \mathbb{R}^3)/(\pi \mathbb{Z} \rtimes_{\mu_d} \mathbb{Z}^3)$, $\mu_d(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}$

e) $(\mathbb{R} \rtimes_{\mu_e} \mathbb{R}^3)/(\mathbb{Z} \rtimes_{\mu_e} \mathbb{Z}^3)$, $\mu_e(t) = \begin{pmatrix} 1 & t & \frac{1}{2}(t^2 - t) \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

f) Consider $A := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -11 \\ 0 & 1 & 8 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{Z})$. A has $X^3 - 8X^2 + 11X - 1$ as characteristic polynomial which possesses three pairwise different real roots $t_1 \approx 6, 271$, $t_2 \approx 1, 631$ and $t_3 \approx 0, 098$. Therefore, A is conjugate to $\mu_f(1)$, where $\mu_f(t) = \begin{pmatrix} e^{t \ln(t_1)} & 0 & 0 \\ 0 & e^{t \ln(t_2)} & 0 \\ 0 & 0 & e^{t \ln(t_3)} \end{pmatrix}$, and this implies the existence of a lattice Γ_f in the completely solvable Lie group $\mathbb{R} \rtimes_{\mu_f} \mathbb{R}^3$.

g) Let $A := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -8 \\ 0 & 1 & 4 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{Z})$. The characteristic polynomial of A is $X^3 - 4X^2 + 8X - 1$ which has three pairwise different roots $t_1 \approx 0, 134$ and $t_{2,3} = (1/\sqrt{t_1})(\cos(\varphi) \pm i \sin(\varphi)) \approx 1, 933 \pm 1, 935i$. So A is conjugate to $\mu_g(1)$, where $\mu_g(t) = \begin{pmatrix} e^{t \ln(t_1)} & 0 & 0 \\ 0 & e^{t \ln(|t_2|)} \cos(t \varphi) & -e^{t \ln(|t_2|)} \sin(t \varphi) \\ 0 & e^{t \ln(|t_2|)} \sin(t \varphi) & e^{t \ln(|t_2|)} \cos(t \varphi) \end{pmatrix}$, and this implies the existence of a lattice Γ_g in the Lie group $\mathbb{R} \rtimes_{\mu_g} \mathbb{R}^3$.

h) Using Theorem 3.1.1, one shows that

$$\begin{aligned} \gamma_1 &:= \left(1, 1, -\frac{1 + \sqrt{5}}{3 + \sqrt{5}}\right), \\ \gamma_2 &:= \left(-\frac{2(2 + \sqrt{5})}{3 + \sqrt{5}}, \frac{1 + \sqrt{5}}{3 + \sqrt{5}}, -\frac{11 + 5\sqrt{5}}{7 + 3\sqrt{5}}\right), \end{aligned}$$

$$\gamma_3 := (0, 0, \sqrt{5})$$

generate a lattice Γ in $U_3(\mathbb{R})$ with $[\gamma_1, \gamma_2] = \gamma_3$ and γ_3 central.

Define the one-parameter group $\mu_h: \mathbb{R} \rightarrow A(U_3(\mathbb{R}))$ by

$$\mu_h(t)((x, y, z)) = (e^{-tt_1}x, e^{tt_1}y, z),$$

where $t_1 := \ln(\frac{3+\sqrt{5}}{2})$. Then $\mu_h(1)$ preserves the lattice Γ with

$$\mu_h(1)(\gamma_1) = \gamma_1^2 \gamma_2, \quad \mu_h(1)(\gamma_2) = \gamma_1 \gamma_2, \quad \mu_h(1)(\gamma_3) = \gamma_3$$

and therefore, $\mathbb{Z} \rtimes_{\mu_h} \Gamma$ is a lattice in $\mathbb{R} \rtimes_{\mu_h} U_3(\mathbb{R})$.

- i) Consider the Lie group \tilde{G} and the one-parameter group $\tilde{\mu}$ of Case B.2 from the proof of the last theorem. Setting $\gamma_1 = (1, 0, 0)$, $\gamma_2 = (0, 1, 0)$ as well as $\gamma_3 = (0, 0, 1)$, $n = 1$ and $c_1 = c_2 = 0$, one explicitly gets an example.

The manifolds of type c) show that formal spaces with the same minimal model as a Kähler manifold need not be Kählerian. This was proved by Fernández and Gray.

Theorem 3.6.3 ([25]). *Let M be one of the symplectic solvmanifolds of type c) in the last theorem, i.e. M is formal and possesses no complex structure. M has the same minimal model as the Kähler manifold $T^2 \times S^2$. \square*

3.7 Five-dimensional solvmanifolds

We study the five-dimensional solvmanifolds by regarding lattices in the corresponding connected and simply-connected Lie groups. By Proposition 3.2.3, their Lie algebras have to be unimodular. These are listed in Appendix A.

3.7.1 Nilpotent and decomposable solvable Lie algebras

There are nine classes of nilpotent Lie algebras in dimension five, see Table A.2. Each of them has a basis with rational structure constants. By Theorem 3.1.1, the corresponding connected and simply-connected Lie groups admit lattices and accordingly to Theorem 3.1.7, the associated nilmanifolds are formal if and only if they are tori. For $i \in \{4, 5, 6\}$ the connected and simply-connected nilpotent Lie group with Lie algebra $\mathfrak{g}_{5,i}$ possesses the left-invariant contact form x_1 (where x_1 is dual to the basis element $X_1 \in \mathfrak{g}_i$ as in Table A.2). Therefore, the corresponding nilmanifolds are contact.

The eight classes of decomposable unimodular non-nilpotent solvable Lie algebras are listed in Table A.3. Except for $\mathfrak{g}_{4,2} \oplus \mathfrak{g}_1$, the corresponding connected and simply-connected Lie groups admit lattices since both of their factors admit lattices.

Theorem 3.7.1.1. *The connected and simply-connected Lie group $G_{4,2} \times \mathbb{R}$ with Lie algebra $\mathfrak{g}_{4,2} \oplus \mathfrak{g}_1$ possesses no lattice.*

Proof. Write G for $G_{4,2} \times \mathbb{R}$ and

$$\mathfrak{g} = \langle X_1, \dots, X_5 \mid [X_1, X_4] = -2X_1, [X_2, X_4] = X_2, [X_3, X_4] = X_2 + X_3 \rangle$$

for its Lie algebra which has $\mathfrak{n} = \mathbb{R}_{X_1, X_2, X_3, X_5}^4$ as nilradical. Therefore, G can be written as almost abelian Lie group $\mathbb{R} \rtimes_{\mu} \mathbb{R}^4$ with

$$\mu(t) = \exp^{\text{GL}(4, \mathbb{R})}(t \text{ad}(X_4)|_{\mathfrak{n}}) = \begin{pmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{-t} & -te^{-t} & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Corollary 3.3.5, the existence of a lattice in G would imply that there is $t_1 \in \mathbb{R} \setminus \{0\}$ such that $\mu(t_1)$ is conjugate to an element of $\text{SL}(4, \mathbb{Z})$. Clearly, the characteristic polynomial of $\mu(t_1)$ is $P(X) = (X-1)\tilde{P}(X)$, where the polynomial $\tilde{P}(X) = X^3 - kX^2 + lX - 1 \in \mathbb{Z}[X]$ has the double root e^{-t_1} . Lemma B.4 then implies $e^{-t_1} = 1$, i.e. $t_1 = 0$ which is a contradiction. \square

Proposition 3.7.1.2. *If Γ is a lattice in a five-dimensional completely solvable non-nilpotent connected and simply-connected decomposable Lie group G , then G/Γ is formal.*

Proof. Let G, Γ be as in the proposition. As usual, we denote by \mathfrak{g} the Lie algebra of G . We have $\mathfrak{g} = \mathfrak{h} \oplus k\mathfrak{g}_1$ with $k \in \{1, 2\}$ and a certain $(5-k)$ -dimensional completely solvable non-nilpotent Lie algebra \mathfrak{h} , see Tables A.3 and A.1. By complete solvability and Theorem 3.2.11 (ii), G/Γ and the Chevalley-Eilenberg complex of $\mathfrak{h} \oplus k\mathfrak{g}_1$ share their minimal model \mathcal{M} . The lower dimensional discussion above shows that for all \mathfrak{h} which can arise in the decomposition of \mathfrak{g} the algebras $\mathcal{M}_{(\wedge \mathfrak{h}^*, \delta_{\mathfrak{h}})}$ and $\mathcal{M}_{(\wedge k\mathfrak{g}_1^*, \delta=0)} = (\wedge k\mathfrak{g}_1^*, \delta=0)$ are formal. This implies the formality of $\mathcal{M} = \mathcal{M}_{(\wedge \mathfrak{h}^*, \delta_{\mathfrak{h}})} \otimes \mathcal{M}_{(\wedge k\mathfrak{g}_1^*, \delta=0)}$. \square

3.7.2 Indecomposable non-nilpotent Lie algebras

There are 19 classes of indecomposable non-nilpotent Lie algebras in dimension five which are unimodular. These are listed in Tables A.4 – A.7. Instead of the small German letters for the Lie algebras in the mentioned tables, we use capital Latin letters (with the same subscripts) for the corresponding connected and simply-connected Lie groups.

We want to examine which of them admit lattices and where appropriate, whether the quotients are formal. The non-existence proofs of lattices in certain almost abelian Lie groups below are taken from Harshavardhan’s thesis [38]. Some of the existence proofs of lattices in certain almost abelian Lie groups are sketched in [38, pp. 29 and 30].

Almost abelian algebras (with nilradical $\mathfrak{n} := 4\mathfrak{g}_1 = \langle X_1, \dots, X_4 \rangle$)

We now consider the almost abelian Lie groups $G_{5,i} = \mathbb{R} \ltimes_{\mu_i} \mathbb{R}^4$. We write $\mu(t) = \mu_i(t) = \exp^{\text{GL}(4, \mathbb{R})}(t \text{ad}(X_5)|_{\mathfrak{n}})$, where $X_5 \in \mathfrak{g}_{5,i}$ is as in Table A.4 (X_5 depends on i). We know by Corollary 3.3.5, Theorem 3.3.6 and Proposition 3.3.7 that there is a lattice Γ in $G_{5,i}$ if and only if there exists $t_1 \neq 0$ such that $\mu(t_1)$ is conjugate to $\tilde{\mu}(1) \in \text{SL}(4, \mathbb{Z})$ and $\Gamma = \mathbb{Z} \ltimes_{\tilde{\mu}} \mathbb{Z}^4$. This will be used in the proof of the following propositions.

Methods to obtain integer matrices with given characteristic polynomial and necessary conditions for their existence are given in Appendix B.

Proposition 3.7.2.1. *Let $p, q, r \in \mathbb{R}$ with $-1 \leq r \leq q \leq p \leq 1$, $pqr \neq 0$ and $p + q + r = -1$. If the completely solvable Lie group $G_{5,7}^{p,q,r}$ admits a lattice and M denotes the corresponding solvmanifold, then M is formal, $b_1(M) = 1$ and one of the following conditions holds:*

- (i) $b_2(M) = 0$,
- (ii) $b_2(M) = 2$, i.e. $r = -1$, $p = -q \in]0, 1[$ or
- (iii) $b_2(M) = 4$, i.e. $r = q = -1$, $p = 1$.

Moreover, there exist p, q, r as above satisfying (i), (ii) resp. (iii) such that $G_{5,7}^{p,q,r}$ admits a lattice.

Proof. We suppress the sub- and superscripts of G and \mathfrak{g} .

a) Assume, there is a lattice in G and denote the corresponding solvmanifold by M . Since \mathfrak{g} is completely solvable, the inclusion of the Chevalley-Eilenberg complex $(\wedge(x_1, \dots, x_5), \delta)$ into the forms on M induces an isomorphism on cohomology. Moreover, the minimal model of $(\wedge(x_1, \dots, x_5), \delta)$ is isomorphic to the minimal model of M .

δ is given by

$$\delta x_1 = -x_{15}, \quad \delta x_2 = -p x_{25}, \quad \delta x_3 = -q x_{35}, \quad \delta x_4 = -r x_{45}, \quad \delta x_5 = 0.$$

(Here we write x_{ij} for $x_i x_j$.) This implies $b_1(M) = 1$.

One computes the differential of the non-exact generators of degree two in the Chevalley-Eilenberg complex as

$$\begin{aligned} \delta x_{12} &= (1+p) x_{125}, & \delta x_{13} &= (1+q) x_{135}, & \delta x_{14} &= (1+r) x_{145}, \\ \delta x_{23} &= (p+q) x_{235}, & \delta x_{24} &= (p+r) x_{245}, & \delta x_{34} &= (q+r) x_{345}. \end{aligned}$$

$-1 \leq r \leq q \leq p \leq 1$, $pqr \neq 0$ and $p + q + r = -1$ implies $p \neq -1$ and $q \neq -r$ and a short computation yields that either (i), (ii) or (iii) holds.

In each case, we determine the 2-minimal model, i.e. the minimal model up to generators of degree two and will see, that these generators are closed. By

Definition 1.1.4, the minimal model then is 2-formal and Theorem 1.1.6 implies the formality of M .

If we are in case (i), the minimal model has one closed generator of degree one, and no generator of degree two.

If we are in case (ii), we have $r = -1$, $p = -q \in]0, 1[$,

$$\begin{aligned} \delta x_{12} &= (1+p)x_{125} \neq 0, & \delta x_{13} &= (1-p)x_{135} \neq 0, & \delta x_{14} &= 0, \\ \delta x_{23} &= 0, & \delta x_{24} &= (p-1)x_{245} \neq 0, & \delta x_{34} &= (-1-p)x_{345} \neq 0, \end{aligned}$$

$$\begin{aligned} H^1(M, \mathbb{R}) &\cong \langle [x_5] \rangle, \\ H^2(M, \mathbb{R}) &\cong \langle [x_{14}], [x_{23}] \rangle, \end{aligned}$$

and the 2-minimal model $\rho: (\wedge V^{\leq 2}, d) \rightarrow (\wedge(x_1, \dots, x_5), \delta)$ is given by

$$\begin{aligned} \rho(y) &= x_5, & |y| &= 1, & dy &= 0; \\ \rho(z_1) &= x_{14}, & |z_1| &= 2, & dz_1 &= 0; \\ \rho(z_2) &= x_{23}, & |z_2| &= 2, & dz_2 &= 0. \end{aligned}$$

Note, further generators of degree ≤ 2 do not occur, since $y^2 = 0$ (by graded commutativity) and $\rho(yz_i)$ is closed and non-exact. Here we use the construction of the minimal model that we have given in the proof of Theorem 1.1.2.

Case (iii) is similar to case (ii).

b) Now, we show that there are examples for each of the three cases. In case

(i), we follow [38] and consider the matrix $\begin{pmatrix} 1 & 0 & 0 & -2 \\ 1 & 2 & 0 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. It suffices to show

that there are $t_1 \in \mathbb{R} \setminus \{0\}$, $-1 < r < q < p < 1$ with $pqr \neq 0$, $p \neq -q$, $p \neq -r$, $q \neq -r$ and $p + q + r = -1$ such that

$$\mu(t_1) = \exp^{\text{GL}(4, \mathbb{R})}(t_1 \text{ad}(X_5)|_{\mathfrak{n}}) = \begin{pmatrix} e^{-t_1} & 0 & 0 & 0 \\ 0 & e^{-pt_1} & 0 & 0 \\ 0 & 0 & e^{-qt_1} & 0 \\ 0 & 0 & 0 & e^{-rt_1} \end{pmatrix}$$

is conjugate to the matrix above, which has $P(X) = X^4 - 8X^3 + 18X^2 - 10X + 1$ as characteristic polynomial. P has four distinct roots $\lambda_1, \dots, \lambda_4$ with $\lambda_1 \approx 0, 12$, $\lambda_2 \approx 0, 62$, $\lambda_3 \approx 2, 79$ and $\lambda_4 \approx 4, 44$. Define $t_1 := -\ln(\lambda_1)$ and p, q, r by $e^{-pt_1} = \lambda_2$, $e^{-qt_1} = \lambda_3$ and $e^{-rt_1} = \lambda_4$. Then t_1, p, q, r have the desired properties.

In case (ii), regard the matrix $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -23 \\ 0 & 0 & 1 & 10 \end{pmatrix}$ which is conjugate to

$$\mu(t_1) = \begin{pmatrix} e^{-t_1} & 0 & 0 & 0 \\ 0 & e^{-pt_1} & 0 & 0 \\ 0 & 0 & e^{pt_1} & 0 \\ 0 & 0 & 0 & e^{t_1} \end{pmatrix} \text{ for } t_1 = 2 \ln\left(\frac{3+\sqrt{5}}{2}\right) \text{ and } p = \frac{1}{2} \text{ since both}$$

matrices have the same characteristic polynomial which has four distinct real roots.

In case (iii), regard the matrix $\begin{pmatrix} 3 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ which is conjugate to $\mu(t_1) = \begin{pmatrix} e^{-t_1} & 0 & 0 & 0 \\ 0 & e^{-t_1} & 0 & 0 \\ 0 & 0 & e^{t_1} & 0 \\ 0 & 0 & 0 & e^{t_1} \end{pmatrix}$ for $t_1 = \ln(\frac{3+\sqrt{5}}{2})$ since both matrices have the same minimal polynomial by Proposition B.8 (ii). \square

We have seen that a non-formal solvmanifold is a non-toral nilmanifold in dimensions three and four. In higher dimensions this is no longer true as the following proposition shows:

Proposition 3.7.2.2. *The completely solvable Lie group $G_{5.8}^{-1}$ admits a lattice.*

Moreover, for each lattice Γ the corresponding solvmanifold $M = G_{5.8}^{-1}/\Gamma$ has $b_1(M) = 2$ and is not formal.

Proof. Again, we suppress the sub- and superscripts. G admits a lattice since $\mu(t) = \exp^{\text{GL}(4, \mathbb{R})}(t \text{ad}(X_5)|_{\mathfrak{n}}) = \begin{pmatrix} 1 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{pmatrix}$ are conjugated for $t_1 = \ln(\frac{3+\sqrt{5}}{2})$. Note that the transformation matrix $T \in \text{GL}(4, \mathbb{R})$ with $TAT^{-1} = \mu(t_1)$ is

$$T = \begin{pmatrix} 1 & 0 & -1 & -2 \\ \frac{1}{\ln(\frac{3+\sqrt{5}}{2})} & \frac{1}{\ln(\frac{3+\sqrt{5}}{2})} & \frac{1}{\ln(\frac{3+\sqrt{5}}{2})} & \frac{1}{\ln(\frac{3+\sqrt{5}}{2})} \\ -\frac{5+3\sqrt{5}}{10} & -\frac{1}{\sqrt{5}} & \frac{5-3\sqrt{5}}{10} & \frac{3}{2} - \frac{7}{2\sqrt{5}} \\ -\frac{5+3\sqrt{5}}{10} & \frac{1}{\sqrt{5}} & \frac{5+3\sqrt{5}}{10} & \frac{3}{2} + \frac{7}{2\sqrt{5}} \end{pmatrix}.$$

Now, let Γ be an arbitrary lattice in G . By complete solvability and Theorem 3.2.11 (ii), we get the minimal model of $M = G/\Gamma$ as the minimal model \mathcal{M} of the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta)$. The latter is given by

$$\delta x_1 = -x_{25}, \delta x_2 = 0, \delta x_3 = -x_{35}, \delta x_4 = x_{45}, \delta x_5 = 0,$$

which implies $b_1(M) = 2$. Further, the minimal model $\rho: (\bigwedge V, d) \rightarrow (\bigwedge \mathfrak{g}^*, \delta)$ must contain two closed generators y_1, y_2 which map to x_2 and x_5 . Then we have $\rho(y_1 y_2) = x_{25} = -\delta x_1$ and the minimal model's construction in the proof of Theorem 1.1.2 implies that there is another generator u of degree one such that $\rho(u) = -x_1$ and $du = y_1 y_2$. Since $\rho(uy_1) = -x_{12}$ and $\rho(uy_2) = -x_{15}$ are

closed and non-exact, there are no further generators of degree one in V . But this implies that $(u + c)y_1$ is closed and non-exact in \mathcal{M} for each closed element c of degree one. Using the notation of Theorem 1.1.5, we have $u \in N^1, y_1 \in V^1$ and \mathcal{M} is not formal. \square

Proposition 3.7.2.3. *The completely solvable Lie group $G_{5,9}^{p,-2-p}$, $p \geq -1$, does not admit a lattice.*

Proof. The first half of the proof is taken from [38]. Assume there is a lattice.

$\mu(t) = \begin{pmatrix} e^{-t} & -te^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-tp} & 0 \\ 0 & 0 & 0 & e^{t(2+p)} \end{pmatrix}$ is conjugate to an element of $\text{SL}(4, \mathbb{Z})$ for

$t = t_1 \neq 0$ and has roots $e^{-t_1}, e^{-t_1}, e^{-t_1 p}$ and $e^{t_1(2+p)}$. By Proposition B.6, this can occur if and only if $p = -1$. Therefore, for the remainder of the proof we assume $p = -1$.

The Jordan form of $\mu(t_1)$ is $\begin{pmatrix} e^{-t} & 1 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix}$, i.e. the characteristic and

the minimal polynomial of $\mu(t_1)$ are

$$\begin{aligned} P(X) &= (X - e^{-t_1})^2(X - e^{t_1})^2 \\ &= X^4 - 2(e^{-t_1} + e^{t_1})X^3 + (e^{-2t_1} + e^{2t_1} + 4)X^2 - 2(e^{-t_1} + e^{t_1})X + 1, \\ m(X) &= (X - e^{-t_1})^2(X - e^{t_1}) \\ &= X^3 - (2e^{-t_1} + e^{t_1})X^2 + (e^{-2t_1} + 2)X - e^{-t_1}. \end{aligned}$$

Since $\mu(t_1)$ is conjugate to an integer matrix, we have $P(X), m(X) \in \mathbb{Z}[X]$ by Theorem B.3. This is impossible for $t_1 \neq 0$. \square

Proposition 3.7.2.4 ([38]). *The completely solvable Lie group $G_{5,11}^{-3}$ does not admit a lattice.*

Proof. If the group admits a lattice, there exists $t_1 \in \mathbb{R} \setminus \{0\}$ such that

the characteristic polynomial of $\mu(t_1) = \begin{pmatrix} e^{-t_1} & -t_1 e^{-t_1} & \frac{t_1^2}{2} e^{-t_1} & 0 \\ 0 & e^{-t_1} & -t_1 e^{-t_1} & 0 \\ 0 & 0 & e^{-t_1} & 0 \\ 0 & 0 & 0 & e^{3t_1} \end{pmatrix}$ is a

monic integer polynomial with a three-fold root e^{-t_1} and a simple root e^{3t_1} . By Proposition B.6, this is impossible for $t_1 \neq 0$. \square

Proposition 3.7.2.5. *There are $q, r \in \mathbb{R}$ with $-1 \leq q < 0$, $q \neq -\frac{1}{2}$, $r \neq 0$ such that $G_{5,13}^{-1-2q,q,r}$ admits a lattice.*

Proof. We have $\mu_{q,r}(t) = \begin{pmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{t+2qt} & 0 & 0 \\ 0 & 0 & e^{-qt} \cos(rt) & -e^{-qt} \sin(rt) \\ 0 & 0 & e^{-qt} \sin(rt) & e^{-qt} \cos(rt) \end{pmatrix}$ and

as claimed in [38], there exist $t_1 \neq 0$, q_0, r_0 such that $\mu_{q_0, r_0}(t_1)$ is conjugate to $A := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ which implies the existence of a lattice Γ_A in $G_{5,13}^{-1-2q_0, q_0, r_0}$.

If $\lambda_1 \approx 0, 15 < \lambda_2 \approx 3, 47$ denote the real roots and $\lambda_{3,4} \approx 1, 17 \pm i0, 67$ the non-real roots of $P_A(X) = X^4 - 6X^3 + 11X^2 - 8X + 1$, then $t_1 = -\ln(\lambda_1) \approx 1, 86$, $q_0 = \frac{1}{2}(\frac{\ln(\lambda_2)}{t_1} - 1) \approx -0, 16$ and $r_0 = \frac{1}{t_1} \arccos(\operatorname{Re}(\lambda_3)e^{q_0 t_1}) \approx 0, 27$. \square

Remark. If the real number $\frac{\pi}{t_1 r_0}$ is not rational, then Theorems 3.2.11 (iii) and 3.3.8 enable us to show that the manifold $G_{5,13}^{-1-2q_0, q_0, r_0} / \Gamma_A$ has $b_1 = 1$ and is formal.

Proposition 3.7.2.6. *There exists $r \in \mathbb{R} \setminus \{0\}$ such that $G_{5,13}^{-1,0,r}$ admits a lattice.*

Proof. Let $t_1 = \ln(\frac{3+\sqrt{5}}{2})$, $r = \pi/t_1$ and $A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Then A

is conjugate to $\mu_{0,r}(t_1) = \begin{pmatrix} e^{-t_1} & 0 & 0 & 0 \\ 0 & e^{t_1} & 0 & 0 \\ 0 & 0 & \cos(rt_1) & \sin(rt_1) \\ 0 & 0 & \sin(rt_1) & \cos(rt_1) \end{pmatrix}$ and this implies the existence of a lattice.

Note that we have $TAT^{-1} = \mu_{0,r}(t_1)$, where $T = \begin{pmatrix} 1 & \frac{18+8\sqrt{5}}{7+3\sqrt{5}} & 0 & 0 \\ 1 & \frac{2}{3+\sqrt{5}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. \square

Remark. Since the abelianisation of the lattice in the last proof is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2^2$, the constructed solvmanifold has $b_1 = 1$.

Proposition 3.7.2.7. $G_{5,14}^0$ admits a lattice.

Proof. We have $\mu(t) = \begin{pmatrix} 1 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & \cos(t) \end{pmatrix}$. Let $t_1 = \frac{\pi}{3}$, then $\mu(t_1)$

is conjugate to $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, so there is a lattice. Note that the matrix

$$T = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -\frac{\pi}{3} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 1 & -1 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4, \mathbb{R}) \text{ satisfies } TAT^{-1} = \mu(t_1). \quad \square$$

Remark. The abelianisation of the lattice in the last proof is isomorphic to \mathbb{Z}^2 , i.e. the corresponding solvmanifold has $b_1 = 2$.

Proposition 3.7.2.8. *If there is a lattice Γ in the Lie group $G := G_{5.14}^0$ such that $b_1(G/\Gamma) = 2$, then G/Γ is not formal.*

Proof. By Theorem 3.2.11(i), the natural inclusion of the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta) \rightarrow (\Omega(G/\Gamma), d)$ induces an injection on cohomology. $(\bigwedge \mathfrak{g}^*, \delta)$ is given by

$$\delta x_1 = -x_{25}, \delta x_2 = 0, \delta x_3 = -x_{45}, \delta x_4 = x_{35}, \delta x_5 = 0.$$

This implies $b_1(\bigwedge \mathfrak{g}^*, \delta) = 2$, hence $H^1(G/\Gamma, \mathbb{R}) = \langle [x_2], [x_5] \rangle$. Therefore

$$[x_2] \cdot H^1(G/\Gamma, \mathbb{R}) + H^1(G/\Gamma, \mathbb{R}) \cdot [x_5] = \langle [x_{25}] \rangle = \langle [\delta x_1] \rangle = 0,$$

and in the Massey product $\langle [x_2], [x_2], [x_5] \rangle = [-x_{15}]$ is no indeterminacy. Since x_{15} is closed and not exact, G/Γ cannot be formal. \square

Proposition 3.7.2.9. *The completely solvable Lie group $G_{5.15}^{-1}$ admits a lattice. For each lattice the corresponding solvmanifold satisfies $b_1 = 1$ and is non-formal.*

Proof. As we have done above, we suppress the sub- and superscripts. First, we follow [38] and consider the matrix $A := \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ which is con-

jugate to $\mu(t_1) = \begin{pmatrix} e^{-t_1} & -t_1 e^{-t_1} & 0 & 0 \\ 0 & e^{-t_1} & 0 & 0 \\ 0 & 0 & e^{t_1} & -t_1 e^{t_1} \\ 0 & 0 & 0 & e^{t_1} \end{pmatrix}$ for $t_1 = \ln(\frac{3+\sqrt{5}}{2})$. This im-

plies the existence of a lattice. The transformation matrix $T \in \mathrm{GL}(4, \mathbb{R})$ with $TAT^{-1} = \mu(t_1)$ is

$$T = \begin{pmatrix} \frac{2}{5\sqrt{5}} & -\frac{1}{5\sqrt{5}} & -\frac{2}{5\sqrt{5}} & \frac{1}{2} + \frac{3}{50}\sqrt{5} \\ -\frac{3+\sqrt{5}}{10 \ln(\frac{3+\sqrt{5}}{2})} & \frac{2+\sqrt{5}}{5 \ln(\frac{3+\sqrt{5}}{2})} & -\frac{(3+\sqrt{5})^2}{20 \ln(\frac{3+\sqrt{5}}{2})} & \frac{2+\sqrt{5}}{5 \ln(\frac{3+\sqrt{5}}{2})} \\ -\frac{2}{5\sqrt{5}} & \frac{1}{5\sqrt{5}} & \frac{2}{5\sqrt{5}} & \frac{1}{2} - \frac{3}{50}\sqrt{5} \\ -\frac{2}{5(3+\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})} & -\frac{1+\sqrt{5}}{5(3+\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})} & \frac{-3+\sqrt{5}}{5(3+\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})} & -\frac{-1+\sqrt{5}}{5(3+\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})} \end{pmatrix}.$$

Let Γ be an arbitrary lattice in G . By complete solvability and Theorem 3.2.11 (ii), we get the minimal model of $M = G/\Gamma$ as the minimal model \mathcal{M} of the Chevalley-Eilenberg complex $(\wedge \mathfrak{g}^*, \delta)$. The latter is given by

$$\delta x_1 = -x_{15} - x_{25}, \quad \delta x_2 = -x_{25}, \quad \delta x_3 = x_{35} - x_{45}, \quad \delta x_4 = x_{45}, \quad \delta x_5 = 0,$$

which implies $b_1(M) = 1$.

One computes the differential of the non-exact generators of degree two in the Chevalley-Eilenberg complex as

$$\begin{aligned} \delta x_{12} &= 2x_{125}, & \delta x_{13} &= x_{145} + x_{235}, & \delta x_{14} &= x_{245}, \\ \delta x_{23} &= x_{245}, & \delta x_{24} &= 0, & \delta x_{34} &= -2x_{345}. \end{aligned}$$

The minimal model $\rho: (\wedge V, d) \rightarrow (\wedge \mathfrak{g}^*, \delta)$ must contain three closed generators y, z_1, z_2 which map to $x_5, x_{14} - x_{23}$ and x_{24} . We see $\rho(yz_1) = x_{145} - x_{235}$ is closed and non-exact, $\rho(yz_2) = x_{245} = \delta x_{23}$ and the minimal model's construction in the proof of Theorem 1.1.2 implies that there is another generator u of degree two such that $\rho(u) = x_{23}$ and $du = yz_2$. Since $\rho(uy) = x_{235}$ is closed and non-exact, there are no further generators of degree less than or equal to two in V . But this implies that $(u + c)y$ is closed and non-exact in \mathcal{M} for each closed element c of degree two. Using the notation of Theorem 1.1.5, we have $u \in N^2, y \in V^1$ and \mathcal{M} is not formal. \square

Proposition 3.7.2.10 ([38]). $G_{5.16}^{-1,q}, q \neq 0$, does not admit a lattice.

Proof. If the group admits a lattice, there exists $t_1 \in \mathbb{R} \setminus \{0\}$ such that the characteristic polynomial of $\mu(t_1) = \begin{pmatrix} e^{-t_1} & -t_1 e^{-t_1} & 0 & 0 \\ 0 & e^{-t_1} & 0 & 0 \\ 0 & 0 & e^{t_1} \cos(t_1 q) & -e^{t_1} \sin(t_1 q) \\ 0 & 0 & e^{t_1} \sin(t_1 q) & e^{t_1} \cos(t_1 q) \end{pmatrix}$ is a monic integer polynomial with simple roots $e^{t_1}(\cos(t_1 q) \pm i \sin(t_1 q))$ and a double root e^{-t_1} . By Proposition B.6, this is impossible for $t_1 \neq 0$. \square

Proposition 3.7.2.11. There are $p, r \in \mathbb{R}, p \neq 0, r \notin \{0, \pm 1\}$, such that $G_{5.17}^{p,-p,r}$ admits a lattice.

Proof. We follow [38] and consider $A := \begin{pmatrix} 2 & 0 & 0 & -11 \\ 1 & 2 & 0 & -9 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. A is conjugate to $\mu(t_1) = \begin{pmatrix} e^{-t_1 p} \cos(t_1) & -e^{-t_1 p} \sin(t_1) & 0 & 0 \\ e^{-t_1 p} \sin(t_1) & e^{-t_1 p} \cos(t_1) & 0 & 0 \\ 0 & 0 & e^{t_1 p} \cos(t_1 r) & -e^{t_1 p} \sin(t_1 r) \\ 0 & 0 & e^{t_1 p} \sin(t_1 r) & e^{t_1 p} \cos(t_1 r) \end{pmatrix}$ for certain $t_1, p, r \neq 0$, i.e. there is a lattice.

If $\lambda_{1,2} \approx 0,306 \pm i0,025$ and $\lambda_{3,4} \approx 2,694 \pm i1,83$ denote the roots of $P_A(X) = X^4 - 6X^3 + 14X^2 - 7X + 1$, one has $t_1 p = -\ln(|\lambda_1|) \approx 1,181$, hence $t_1 = \arccos(\operatorname{Re}(\lambda_1)e^{t_1 p}) \approx 0,062$, $p \approx 14,361$. $t_1 r = \arccos(\operatorname{Re}(\lambda_3)e^{t_1 p}) \approx 0,597$ implies $r \approx 7,259$. \square

Remark. Since the abelianisation of the lattice in the last proof is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$, the corresponding solvmanifold has $b_1 = 1$.

Proposition 3.7.2.12. *There exists $p \in \mathbb{R} \setminus \{0\}$ such that $G_{5,17}^{p,-p,\pm 1}$ admits a lattice.*

Proof. Let $p := \frac{1}{\pi} \ln(\frac{3+\sqrt{5}}{2})$, $t_1 := \pi$ and $A := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{pmatrix}$. Then $\mu(t_1) = \begin{pmatrix} e^{-t_1 p} \cos(t_1) & -e^{-t_1 p} \sin(t_1) & 0 & 0 \\ e^{-t_1 p} \sin(t_1) & e^{-t_1 p} \cos(t_1) & 0 & 0 \\ 0 & 0 & e^{t_1 p} \cos(\pm t_1) & -e^{t_1 p} \sin(\pm t_1) \\ 0 & 0 & e^{t_1 p} \sin(\pm t_1) & e^{t_1 p} \cos(\pm t_1) \end{pmatrix}$ is conjugate to A and this implies the existence of a lattice. Note that we have $TAT^{-1} = \mu(t_1)$ with $T := \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{5-3\sqrt{5}}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}} & \frac{5-3\sqrt{5}}{10} \\ -\frac{1}{\sqrt{5}} & \frac{5+3\sqrt{5}}{10} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{5+3\sqrt{5}}{10} \end{pmatrix}$. \square

Remark. The abelianisation of the lattice in the last proof is $\mathbb{Z} \oplus \mathbb{Z}_3^2$, i.e. the corresponding solvmanifold has $b_1 = 1$.

Proposition 3.7.2.13. *There exists $r \in \mathbb{R} \setminus \{0, \pm 1\}$ such that $G_{5,17}^{0,0,r}$ admits a lattice.*

Proof. Let $r \in \{2, 3\}$. Then $\mu(t) = \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(tr) & \sin(tr) \\ 0 & 0 & \sin(tr) & \cos(tr) \end{pmatrix}$ is an integer matrix for $t = \pi$. This implies the existence of a lattice. \square

Remark. If we chose in the last proof $r = 2$, then the corresponding solvmanifold has $b_1 = 3$. For $r = 3$ we obtain a solvmanifold with $b_1 = 1$.

Proposition 3.7.2.14. $G_{5,17}^{0,0,\pm 1}$ admits a lattice.

Proof. $\mu(t) = \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(\pm t) & \sin(\pm t) \\ 0 & 0 & \sin(\pm t) & \cos(\pm t) \end{pmatrix}$ is an integer matrix for $t = \pi$. This implies the existence of a lattice. \square

Remark. The first Betti number of the solvmanifold induced by the lattice of the last proof equals one.

Proposition 3.7.2.15. $G_{5.18}^0$ admits a lattice.

Proof. Again, we follow [38]. The matrix $\begin{pmatrix} 2 & 0 & 0 & -9 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is conjugate to $\mu(t_1) = \begin{pmatrix} \cos(t_1) & -\sin(t_1) & -t_1 \cos(t_1) & t_1 \sin(t_1) \\ \sin(t_1) & \cos(t_1) & -t_1 \sin(t_1) & -t_1 \cos(t_1) \\ 0 & 0 & \cos(t_1) & -\sin(t_1) \\ 0 & 0 & \sin(t_1) & \cos(t_1) \end{pmatrix}$ for $t_1 = \frac{\pi}{3}$. This implies the existence of a lattice.

Note, $T = \begin{pmatrix} \frac{4}{3\sqrt{3}} & -\frac{2}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & 1 \\ \frac{\sqrt{3}}{\pi} & -\frac{2\sqrt{3}}{\pi} & -\frac{\sqrt{3}}{\pi} & \frac{\sqrt{3}}{\pi} \\ \frac{1}{\pi} & 0 & -\frac{3}{\pi} & -\frac{3}{\pi} \end{pmatrix} \in \text{GL}(4, \mathbb{R})$ is the transformation matrix with $TAT^{-1} = \mu(t_1)$. \square

Remark. The abelianisation of the lattice in the last proof is isomorphic to \mathbb{Z} , i.e. the corresponding solvmanifold has $b_1 = 1$.

Algebras with nilradical $\mathfrak{n} := \mathfrak{g}_{3.1} \oplus \mathfrak{g}_1 = \langle X_1, \dots, X_4 \mid [X_2, X_3] = X_1 \rangle$

We now regard the unimodular almost-nilpotent Lie groups $G_{5.i}$ with nilradical $N := U_3(\mathbb{R}) \times \mathbb{R}$, i.e. $i \in \{19, 20, 23, 25, 26, 28\}$. We can identify N with \mathbb{R}^4 as a manifold and the group law given by

$$(a, b, c, r) \cdot (x, y, z, w) = (a + x + bz, b + y, c + z, r + w).$$

The Lie algebras of the unimodular Lie groups $G_{5.i} = \mathbb{R} \ltimes_{\mu_i} N$ with nilradical N are listed in Table A.5. We have $\mu_i(t) = \exp^N \circ \exp^{A(\mathfrak{n})}(t \text{ad}(X_5)) \circ \log^N$, where X_5 depends on i .

Assume there is a lattice Γ in $G_{5.i}$. By Corollary 3.3.5, there are $t_1 \neq 0$ and an inner automorphism I_{n_1} of N such that $\nu_i := \mu_i(t_1) \circ I_{n_1}, \nu_i^{-1} \in A(N)$ preserve the lattice $\Gamma_N := \Gamma \cap N$ in N . For $n_1 = (a, b, c, r)$ one calculates

$$I_{n_1}(x, y, z, w) = (x + bz - yc, y, z, w). \quad (3.8)$$

$\Gamma_{N'} := \Gamma_N \cap N' \cong \mathbb{Z}$ is a lattice in $N' := [N, N] = \{(x, 0, 0, 0) \mid x \in \mathbb{R}\} \cong \mathbb{R}$ by Theorem 3.1.4 and since $\nu_i(\Gamma_{N'}), \nu_i^{-1}(\Gamma_{N'}) \subset \Gamma_{N'}$, we have $\nu_i|_{\Gamma_{N'}} \in \text{Aut}(\mathbb{Z})$. This implies $\nu_i|_{\Gamma_{N'}} = \pm \text{id}$ and hence $\mu_i(t_1)|_{[N, N]} = \pm \text{id}$ (a cause of (3.8) and the shape of $[N, N]$). Moreover, we have $[\mathfrak{n}, \mathfrak{n}] = \langle X_1 \rangle$ and since $\exp^{\mathbb{R}}$ is the identity,

$$\pm \text{id} = \mu_i(t_1)|_{[N, N]} = \exp^{A(\mathfrak{n})}(t_1 \text{ad}(X_5)|_{\mathfrak{n}})|_{[N, N]}.$$

(Note that $\exp^N([\mathfrak{n}, \mathfrak{n}]) = [N, N]$ by [76, Theorem 3.6.2].) Therefore, $t_1[X_5, X_1]$ has no component in $\langle X_1 \rangle$ and since $t_1 \neq 0$, this means that $[X_1, X_5]$ has no component in X_1 -direction. The list of Lie algebras in Table A.5 implies:

Proposition 3.7.2.16. *The only connected and simply-connected solvable Lie groups with nilradical $U_3(\mathbb{R}) \times \mathbb{R}$ that can contain a lattice are $G_{5.20}^{-1}$ and $G_{5.26}^{0, \pm 1}$. \square*

Proposition 3.7.2.17. *$G_{5.20}^{-1}$ admits a lattice. For each lattice the corresponding solvmanifold admits a contact form, is formal and has $b_1 = 2$.*

Proof. Using Theorem 3.1.1, one shows that

$$\begin{aligned}\gamma_1 &:= \left(\frac{20 + 9\sqrt{5}}{9 + 4\sqrt{5}}, 0, 0, 0 \right), \\ \gamma_2 &:= \left(\frac{181 + 81\sqrt{5}}{47 + 21\sqrt{5}}, \frac{18 + 8\sqrt{5}}{7 + 3\sqrt{5}}, \frac{2}{3 + \sqrt{5}}, 0 \right), \\ \gamma_3 &:= \left(\frac{181 + 81\sqrt{5}}{47 + 21\sqrt{5}}, 1, 1, 0 \right), \\ \gamma_4 &:= \left(0, 0, 0, -\frac{20 + 9\sqrt{5}}{(9 + 4\sqrt{5}) \ln\left(\frac{3+\sqrt{5}}{2}\right)} \right)\end{aligned}$$

generate a lattice Γ_N in N with $[\gamma_2, \gamma_3] = \gamma_1$ and γ_1, γ_4 central.

A short calculation yields that $\mu(t)((x, y, z, w)) = (x - tw, e^{-t}y, e^t z, w)$ defines a one-parameter group in $A(N)$. Moreover, for $t_1 = \ln\left(\frac{3+\sqrt{5}}{2}\right)$ holds $\mu(t_1)(\gamma_1) = \gamma_1$, $\mu(t_1)(\gamma_2) = \gamma_3$, $\mu(t_1)(\gamma_3) = \gamma_2^{-1}\gamma_3^3$ and $\mu(t_1)(\gamma_4) = \gamma_1\gamma_4$.

This implies the existence of a lattice in $G := G_{5.20}^{-1} = \mathbb{R} \ltimes_{\mu} N$.

Let Γ be an arbitrary lattice in G . By complete solvability and Theorem 3.2.11 (ii), we get the minimal model of $M = G/\Gamma$ as the minimal model \mathcal{M} of the Chevalley-Eilenberg complex $(\wedge \mathfrak{g}^*, \delta)$. The latter is given by

$$\delta x_1 = -x_{23} - x_{45}, \quad \delta x_2 = -x_{25}, \quad \delta x_3 = x_{35}, \quad \delta x_4 = \delta x_5 = 0,$$

which implies $b_1(M) = 2$. Moreover, x_1 defines a left-invariant contact form on G/Γ .

One computes the differential of the non-exact generators of degree two in the Chevalley-Eilenberg complex as

$$\begin{aligned}\delta x_{12} &= x_{125} - x_{245}, & \delta x_{13} &= -x_{135} - x_{345}, & \delta x_{14} &= -x_{234}, \\ \delta x_{15} &= -x_{235}, & \delta x_{23} &= 0, & \delta x_{24} &= x_{245}, \\ \delta x_{34} &= -x_{345}, & \delta x_{45} &= 0.\end{aligned}$$

The minimal model $\rho: (\wedge V, d) \rightarrow (\wedge \mathfrak{g}^*, \delta)$ must contain two closed generators y_1, y_2 which map to x_4 and x_5 . We see $\rho(y_1 y_2) = x_{45}$ is closed and non-exact. Since $b_2(G/\Gamma) = 1$, the minimal model's construction in the proof of Theorem 1.1.2 implies that there are no further generators of degree less than or equal to two in V . This implies that G/Γ is formal. \square

Proposition 3.7.2.18. $G_{5,26}^{0,\varepsilon}$ admits a lattice for $\varepsilon = \pm 1$. For each lattice the corresponding solvmanifold is contact and has $b_1 \geq 2$.

Proof. One calculates that $\mu: \mathbb{R} \rightarrow A(N)$ defined by

$$\begin{aligned} \mu(t)((x, y, z, w)) \\ = (x + h_t(y, z) - \varepsilon tw, \cos(t\pi)y - \sin(t\pi)z, \sin(t\pi)y + \cos(t\pi)z, w), \end{aligned}$$

where $h_t(y, z) = \frac{1}{2} \sin(t\pi) \left(\cos(t\pi)(y^2 - z^2) - 2 \sin(t\pi)yz \right)$, is a one-parameter group.

Then we have $G := G_{5,26}^{0,\varepsilon} = \mathbb{R} \ltimes_{\mu} N$ and $\mathbb{Z} \ltimes_{\mu} \{(x, y, z, w) \in N \mid x, y, z, w \in \mathbb{Z}\}$ is a lattice in G since $\mu(1)((x, y, z, w)) = (x - \varepsilon w, -y, -z, w)$.

Using $d_e(\mu(t)) = \log^N \circ \mu(t) \circ \exp^N$, we obtain the Lie algebra \mathfrak{g} of G as

$$\langle X_1, \dots, X_5 \mid [X_2, X_3] = X_1, [X_2, X_5] = X_3, [X_3, X_5] = -X_2, [X_4, X_5] = \varepsilon X_1 \rangle.$$

Denote $\{x_1, \dots, x_5\}$ the basis of \mathfrak{g}^* which is dual to $\{X_1, \dots, X_5\}$, i.e. the x_i are left-invariant 1-forms on G . One calculates that x_1 is a left-invariant contact form on G , so it descends to a contact form on the corresponding solvmanifold.

The statement about the first Betti number follows from Theorem 3.2.11(i). \square

Remark. Since the abelianisation of the lattice in the last proof is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$, the corresponding solvmanifold has $b_1 = 2$.

Algebras with nilradical $\mathfrak{g}_{4,1} = \langle X_1, \dots, X_4 \mid [X_2, X_4] = X_1, [X_3, X_4] = X_2 \rangle$

Proposition 3.7.2.19. No connected and simply-connected solvable Lie group $G_{5,i}$ with nilradical $N := G_{4,1}$ admits a lattice.

Proof. There is only one unimodular connected and simply-connected solvable Lie group with nilradical $G_{4,1}$, namely the completely solvable group $G := G_{5,30}^{-\frac{4}{3}}$. We show that it admits no lattice.

The group N is \mathbb{R}^4 as a manifold with multiplication given by

$$(a, b, c, r) \cdot (x, y, z, w) = (a + x + wb + \frac{1}{2}w^2c, b + y + wc, c + z, r + w),$$

and one calculates for $n_1 = (a, b, c, r)$

$$I_{n_1}(x, y, z, w) = (x + wb + \frac{1}{2}w^2c - ry - rwc + \frac{1}{2}r^2z, y + wc - rz, z, w).$$

Let $G = \mathbb{R} \ltimes_{\mu} N$, where $\mu(t) = \exp^N \circ \exp^{A(n)}(t \operatorname{ad}(X_5)|_n) \circ \log^N$ and assume there is a lattice Γ in G . By Corollary 3.3.5, there are $t_1 \neq 0$ and $n_1 \in N$ such that $\nu := \mu(t_1) \circ I_{n_1} \in A(N)$ preserves the lattice $\Gamma_N := \Gamma \cap N$ in N .

$\Gamma_{N'} := N' \cap \Gamma_N$ is a lattice in $N' := [N, N] = \{(x, y, 0, 0) \in N \mid x, y \in \mathbb{R}\} \cong \mathbb{R}^2$ by Theorem 3.1.4, and since $\nu(N') \subset N'$, this lattice is preserved by $\nu|_{N'}$. This and $\exp^{\mathbb{R}^2} = \text{id}$ imply

$$\pm 1 = \det(\nu|_{N'}) = \det \left(\exp^{A(\mathfrak{n})}(t_1 \text{ad}(X_5)|_{\mathfrak{n}})|_{[N, N]} \right) \cdot \underbrace{\det(I_{n_1}|_{N'})}_{= 1},$$

i.e. $\text{ad}(X_5)|_{[\mathfrak{n}, \mathfrak{n}]}$ has trace equal to zero. This and $[\mathfrak{n}, \mathfrak{n}] = \langle X_1, X_2 \mid \rangle$ contradicts $\mathfrak{g}_{5,30}^{-\frac{4}{3}}$, see Table A.6. \square

Non-almost nilpotent algebras

Now, there remain two unimodular connected and simply-connected solvable Lie groups in dimension five, namely $G_{5,33}^{-1,-1}$ and $G_{5,35}^{-2,0}$. Unfortunately, we do not know whether the former group admits a lattice or not. Note, Harshavardhan’s argumentation in [38, p. 33] is not sufficient.

Remark. If the completely solvable Lie group $G_{5,33}^{-1,-1}$ admits a lattice, one easily proves that the corresponding solvmanifold admits a contact form (since $G_{5,33}^{-1,-1}$ possesses the left-invariant contact form $x_1 + x_2 + x_3$ with x_i dual to $X_i \in \mathfrak{g}_{5,33}^{-1,-1}$ as in Table A.7), is formal and has $b_1 = 2$.

Remark. In April 2009, A. Diatta and B. Foreman proved that $G_{5,33}^{-1,-1}$ possesses a lattice.

Proposition 3.7.2.20. $G_{5,35}^{-2,0}$ contains a lattice. For each lattice the corresponding solvmanifold is contact and has $b_1 \geq 2$.

Proof. A lattice and a contact form were constructed by Geiges in [32]. One has the left-invariant contact form $x_1 + x_2$ on the Lie group, where x_1, x_2 are dual to the left-invariant vector fields as in Table A.7. Hence the form descends to each compact quotient by a discrete subgroup.

The statement about the first Betti number follows from Theorem 3.2.11(i). \square

Conclusion

We have seen that each connected and simply-connected 5-dimensional solvable Lie group admits a lattice if it is nilpotent or decomposable with the exception of $G_{4,2} \times \mathbb{R}$. If an indecomposable non-nilpotent group $G_{5,i}$ gives rise to a solvmanifold it is contained in Table 3.3. Recall, by Theorem 3.2.11, we always have a lower bound for the solvmanifold’s Betti numbers and in some cases the exact value. These can be read of in the second and the third column. The last column refers to the examples that we have constructed above. “yes” means that we have such for certain parameters that satisfy the conditions of the column “Comment”. Except for $i = 33$ we have examples for all possible values of i .

Table 3.3: 5-dimensional indecomposable non-nilmanifolds

	b_1	b_2	formal	Comment	Example
$G_{5.7}^{p,q,r}$	1	0	yes	$-1 < r < p < q < 1,$ $pqr \neq 0,$ $p + q + r = -1$	3.7.2.1 (i)
$G_{5.7}^{p,q,-1}$	1	2	yes	$p = -q \in]0, 1[$	3.7.2.1 (ii)
$G_{5.7}^{1,-1,-1}$	1	4	yes		3.7.2.1 (iii)
$G_{5.8}^{-1}$	2	3	no		3.7.2.2
$G_{5.13}^{-1-2q,q,r}$	≥ 1	≥ 0	?	$q \in [-1, 0] \setminus \{\frac{1}{2}\},$ $r \neq 0$	3.7.2.5
$G_{5.13}^{-1,0,r}$	≥ 1	≥ 2	?	$r \neq 0$	3.7.2.6
$G_{5.14}^0$	≥ 2	≥ 3	?		3.7.2.7
$G_{5.15}^{-1}$	1	2	no		3.7.2.9
$G_{5.17}^{p,-p,r}$	≥ 1	≥ 0	?	$p \neq 0, r \notin \{0, \pm 1\}$	3.7.2.11
$G_{5.17}^{p,-p,\pm 1}$	≥ 1	≥ 2	?	$p \neq 0$	3.7.2.12
$G_{5.17}^{0,0,r}$	≥ 1	≥ 2	?	$r \notin \{0, \pm 1\}$	3.7.2.13
$G_{5.17}^{0,0,\pm 1}$	≥ 1	≥ 4	?		3.7.2.14
$G_{5.18}^0$	≥ 1	≥ 2	?		3.7.2.15
$G_{5.20}^{-1}$	2	1	yes		3.7.2.17
$G_{5.26}^{0,\pm 1}$	≥ 2	≥ 1	?		3.7.2.18
$G_{5.33}^{-1,-1}$	2	1	yes		no
$G_{5.35}^{-2,0}$	≥ 2	≥ 1	?		3.7.2.20

Assuming that there is a lattice in one the non-completely solvable Lie groups $G_{5,i}$, i.e. $i \in \{13, 14, 17, 18, 26, 35\}$, such that the inequalities in the table on page 64 are equalities, then one can calculate that such quotients are formal for $i \in \{13, 17, 26, 35\}$ and not formal for $i \in \{14, 18\}$. The assumptions about the Betti numbers are needed to ensure that the Lie algebra cohomology is isomorphic to the solvmanifold's cohomology.

3.7.3 Contact structures

Some of the connected and simply-connected five-dimensional solvable Lie groups $G_{5,i}$ which admit a lattice Γ possess a left-invariant contact form. Obviously, it also defines a contact form on the corresponding solvmanifold. By this way, we showed that the manifolds $G_{5,i}/\Gamma$ for $i \in \{4, 5, 6\}$ and quotients of almost nilpotent groups with non-abelian nilradical (i.e. $i \geq 19$) by lattices are contact.

But \mathbb{R}^5 , $U_3(\mathbb{R}) \times \mathbb{R}^2$, $G_{4.1} \times \mathbb{R}$ and $G_{5,i}$ do not have a left-invariant contact form for $i \in \{1, 2, 3, 7, \dots, 18\}$, see e.g. [17]. For some of the nilmanifolds, we can provide a contact structure by another approach.

Theorem 3.7.3.1. *Let $G \in \{\mathbb{R}^5, U_3(\mathbb{R}) \times \mathbb{R}^2, G_{4.1} \times \mathbb{R}, G_{5.1}, G_{5.3}\}$ and Γ a lattice G . Then G/Γ admits a contact structure.*

Proof. For G chosen as in the theorem, the dimension of the center is greater than or equal to two. Therefore, we can find a two-dimensional closed normal subgroup that lies in the center such that its intersection with Γ is a lattice in it. By Theorem 3.2.6, G/Γ has the structure of a principal T^2 -bundle over a three dimensional closed orientable manifold. Then the following result of Lutz implies the claim. \square

Theorem 3.7.3.2 ([51]). *The total space of a principal T^2 -bundle over a closed orientable 3-manifold admits a contact form.* \square

Unfortunately, we did not find a contact structure on the manifold of Proposition 3.7.2.9. If such exists, this yields a five-dimensional non-formal contact solvmanifold with $b_1 = 1$.

3.8 Six-dimensional solvmanifolds

There are 164 types of connected and simply-connected indecomposable solvable Lie groups in dimension six, most of them depending on parameters. For classifying six-dimensional solvmanifolds, we restrict ourselves to the following types:

- (a) nilmanifolds,
- (b) symplectic solvmanifolds that are quotients of indecomposable groups which are not nilpotent,
- (c) products of lower-dimensional solvmanifolds.

Although we have to make some restrictions to get a manageable number of cases, one certainly has to consider types (a) and (c). Concerning the third type, the reader can even ask the legitimate question why we do not consider arbitrary lattices in products of lower dimensional solvable Lie groups G_1, G_2 , instead of direct products $\Gamma_1 \times \Gamma_2$ of lattices Γ_i in the factors G_i . The reason is that we have no tool to construct arbitrary lattices or disprove their existence, unless we can ensure that they contain the semidirect factor \mathbb{Z} . (When we wanted to investigate $G_{5.33}^{-1,-1}$, we already had this problem.)

The further restriction in (b) is justified by the large number of indecomposable non-nilpotent solvable Lie algebras in dimension six: There are 140 types of it. The author has decided to consider the most interesting among them. Since we are not able to refute a symplectic form's existence in the non-completely solvable case, we shall partly make even more restrictions.

3.8.1 Nilmanifolds

There are 34 isomorphism classes of nilpotent Lie algebras in dimension six. Each of them possesses a basis with rational structure constants and therefore determines a nilmanifold. They are listed on page 67 in Table 3.4 which is taken from [68]. The corresponding Lie algebras are listed in Appendix A. Among the 34 classes of nilmanifolds, there are 26 which admit a symplectic form.

Recall that a nilmanifold is formal or Kählerian if and only if the corresponding Lie algebra is abelian.

3.8.2 Candidates for the existence of lattices

Among the 61 types of indecomposable unimodular almost nilpotent Lie algebras in dimension six that are listed in Tables A.10 – A.23, there are some that cannot be the Lie algebra of a connected and simply-connected Lie group which admits a lattice.

Instead of the small German letters for the Lie algebras in the mentioned tables, we use again capital Latin letters with the same subscripts for the corresponding connected and simply-connected Lie groups. If any, we chose the same designation for the parameters $a, b, c, h, s, \varepsilon$ of $G_{6,i}$ as for their Lie algebras.

Proposition 3.8.2.1. *Let $i \in \{13, \dots, 38\}$, i.e. $\text{Nil}(G_{6,i}) = U_3(\mathbb{R}) \times \mathbb{R}^2$. Then it is necessary for $G_{6,i}$ to contain a lattice that one of the following conditions holds:*

$$\begin{array}{lll}
 i = 13, a = -b \neq 0, h = -1; & i = 15; & i = 18, a = -1; \\
 i = 21, a = 0; & i = 23, a = 0; & i = 25, b = 0; \\
 i = 26; & i = 29, b = 0; & i = 32, a = \varepsilon = 0 < h; \\
 i = 33, a = 0; & i = 34, a = 0; & i = 35, a = -b \neq 0, c = 0; \\
 i = 36, a = 0; & i = 37, a = 0, s \neq 0; & i = 38.
 \end{array}$$

Proof. This can be seen analogous as in the proof of Proposition 3.7.2.16. Denote $\{X_1, \dots, X_6\}$ the basis used for the description of the Lie algebra in Tables A.12 – A.14. Then the existence of a lattice implies that $[X_6, X_1]$ has no component in X_1 -direction and this yields the claim. \square

Proposition 3.8.2.2. *Let $i \in \{39, \dots, 47\}$, i.e. the nilradical of $G_{6,i}$ is $G_{4.1} \times \mathbb{R}$. If $G_{6,i}$ admits a lattice, then holds $i = 39 \wedge h = -3$ or $i = 40$.*

Proof. Use the designation X_1, \dots, X_6 as above. Then $\langle X_1, X_2 \rangle$ is the commutator of the nilradical of $\mathfrak{g}_{6,i}$. Analogous as in the proof of Proposition 3.7.2.19, one shows that $\text{ad}(X_6)|_{\langle X_1, X_2 \rangle}$ has trace equal to zero. This is only satisfied for $i = 39 \wedge h = -3$ or $i = 40$. \square

Table 3.4: 6-dimensional nilmanifolds

$b_1(G/\Gamma)$	$b_2(G/\Gamma)$	Comment	\mathfrak{g}
6	15	Torus, symplectic	$6\mathfrak{g}_1$
5	11	symplectic	$\mathfrak{g}_{3.1} \oplus 3\mathfrak{g}_1$
5	9	not symplectic	$\mathfrak{g}_{5.4} \oplus \mathfrak{g}_1$
4	9	symplectic	$\mathfrak{g}_{5.1} \oplus \mathfrak{g}_1$
4	8	symplectic	$2\mathfrak{g}_{3.1}$
4	8	symplectic	$\mathfrak{g}_{6.N4}$
4	8	symplectic	$\mathfrak{g}_{6.N5}$
4	7	symplectic	$\mathfrak{g}_{5.5} \oplus \mathfrak{g}_1$
4	7	symplectic	$\mathfrak{g}_{4.1} \oplus 2\mathfrak{g}_1$
4	6	not symplectic	$\mathfrak{g}_{6.N12}$
3	8	symplectic	$\mathfrak{g}_{6.N3}$
3	6	symplectic	$\mathfrak{g}_{6.N1}$
3	6	symplectic	$\mathfrak{g}_{6.N6}$
3	6	symplectic	$\mathfrak{g}_{6.N7}$
3	5	symplectic	$\mathfrak{g}_{5.2} \oplus \mathfrak{g}_1$
3	5	not symplectic	$\mathfrak{g}_{5.3} \oplus \mathfrak{g}_1$
3	5	symplectic	$\mathfrak{g}_{5.6} \oplus \mathfrak{g}_1$
3	5	symplectic	$\mathfrak{g}_{6.N8}$
3	5	symplectic	$\mathfrak{g}_{6.N9}$
3	5	symplectic	$\mathfrak{g}_{6.N10}$
3	5	not symplectic	$\mathfrak{g}_{6.N13}$
3	5	not symplectic	$\mathfrak{g}_{6.N14}^1$
3	5	not symplectic	$\mathfrak{g}_{6.N14}^{-1}$
3	5	symplectic	$\mathfrak{g}_{6.N15}$
3	5	symplectic	$\mathfrak{g}_{6.N17}$
3	4	symplectic	$\mathfrak{g}_{6.N16}$
2	4	symplectic	$\mathfrak{g}_{6.N11}$
2	4	symplectic	$\mathfrak{g}_{6.N18}^1$
2	4	symplectic	$\mathfrak{g}_{6.N18}^{-1}$
2	3	symplectic	$\mathfrak{g}_{6.N2}$
2	3	symplectic	$\mathfrak{g}_{6.N19}$
2	3	symplectic	$\mathfrak{g}_{6.N20}$
2	2	not symplectic	$\mathfrak{g}_{6.N21}$
2	2	not symplectic	$\mathfrak{g}_{6.N22}$

Proposition 3.8.2.3.

- (i) Let $i \in \{54, \dots, 70\}$, i.e. the nilradical of $G_{6,i}$ is $G_{5,1}$. If $G_{6,i}$ admits a lattice, then holds $i = 54 \wedge l = -1$, $i = 63$, $i = 65 \wedge l = 0$ or $i = 70 \wedge p = 0$.
- (ii) No connected and simply-connected almost nilpotent Lie group with nilradical $G_{5,2}$ or $G_{5,5}$ admits a lattice. \square

Proof. This follows in the same manner as the last proposition. The trace of $\text{ad}(X_6)$ restricted to the commutator of the nilradical must be zero. \square

3.8.3 Symplectic solvmanifolds whose first Betti number equals one

If we are looking for solvmanifolds with $b_1 = 1$, it is necessary that the corresponding Lie algebra is unimodular, almost nilpotent and has $b_1 = 1$ itself. Note that the latter forces the algebra to be indecomposable. In Tables A.27 – A.29 on pages 115 – 117 we have listed all possible values that can arise as b_1 for the classes of unimodular non-nilpotent solvable indecomposable Lie algebras in dimension six.

Since we are mainly interested in symplectic 6-manifolds, we now investigate which Lie algebras contained in Tables A.10 – A.23 that satisfy $b_1 = 1$ are *cohomologically symplectic*, i.e. there is a closed element $\omega \in \bigwedge^2 \mathfrak{g}^*$ such that ω^3 is not exact.

Note, if a unimodular Lie algebra is cohomologically symplectic, then each compact quotient of the corresponding Lie group by a lattice is symplectic. If the Lie algebra is completely solvable, this is even necessary for the quotient to be symplectic.

Proposition 3.8.3.1. *Let $\mathfrak{g}_{6,i}$ be a unimodular almost-nilpotent Lie algebra with $b_1(\mathfrak{g}_{6,i}) = 1$. Then we have:*

$\mathfrak{g}_{6,i}$ is cohomologically symplectic if and only if $i \in \{15, 38, 78\}$.

Proof. For $i \in \{15, 38, 78\}$ one computes all symplectic forms up to exact summands as

$$i = 15 : \omega = (\lambda + \mu)x_{16} + \lambda x_{25} - \mu x_{34}, \quad \lambda, \mu \in \mathbb{R} \setminus \{0\}, \lambda \neq -\mu,$$

$$i = 38 : \omega = \lambda x_{16} + \mu x_{24} + \frac{\lambda}{2} x_{25} - \frac{\lambda}{2} x_{34} + \mu x_{35}, \quad \lambda, \mu \in \mathbb{R}, \lambda \neq 0, -\frac{3}{2}\lambda^3 \neq 2\lambda\mu^2,$$

$$i = 78 : \omega = \lambda x_{14} + \lambda x_{26} + \lambda x_{35}, \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

If $i \notin \{15, 38, 78\}$, then the conditions on the parameters of $\mathfrak{g}_{6,i}$ to ensure its unimodularity and $b_1(\mathfrak{g}_{6,i}) = 1$ imply that there are no closed elements of $\bigwedge^2 \mathfrak{g}_{6,i}^*$ without exact summands which contain one of the elements $x_{16}, x_{26}, x_{36}, x_{46}$ or x_{56} . Therefore, $\mathfrak{g}_{6,i}$ cannot be cohomologically symplectic. \square

Remark. We give an explicit example of the argumentation in the last proof for $i = 2$:

$\mathfrak{g}_{6,2}$ depends on three parameters $a, c, d \in \mathbb{R}$ with $0 < |d| \leq |c| \leq 1$ and the brackets are given in Table A.11 as

$$\begin{aligned} [X_1, X_6] &= a X_1, & [X_2, X_6] &= X_1 + a X_2, & [X_3, X_6] &= X_3, \\ [X_4, X_6] &= c X_4, & [X_5, X_6] &= d X_5. \end{aligned}$$

The condition of unimodularity implies $2a + c + d = -1$. Moreover, if first the Betti number equals one, we see in Tabular A.27 that $a \neq 0$.

The Chevalley-Eilenberg complex is given by

$$\begin{aligned} \delta x_1 &= -a x_{16} - x_{26}, & \delta x_2 &= -a x_{26}, & \delta x_3 &= -x_{36}, \\ \delta x_4 &= -c x_{46}, & \delta x_5 &= -d x_{56}, & \delta x_6 &= 0 \end{aligned}$$

and since $a, c, d \neq 0$, $x_{26}, x_{36}, x_{46}, x_{56}$ are exact. Moreover, $x_{16} = \delta(-\frac{1}{a}x_1 + \frac{1}{a^2}x_2)$ is exact, too. This implies the claim.

We now examine the three Lie groups that have cohomologically symplectic Lie algebras.

The next theorem was announced in Chapter 2. It provides an example of a symplectic non-formal 6-manifold with $b_1 = 1$. Since it is a solvmanifold, this manifold is symplectically aspherical. Hence, we found an example for which Kędra, Rudyak and Tralle looked in [48, Remark 6.5].

Theorem 3.8.3.2.

- (i) *The completely solvable Lie group $G_{6,15}^{-1}$ contains a lattice.*
- (ii) *If Γ is any lattice in $G := G_{6,15}^{-1}$, then $M := G/\Gamma$ is a symplectic and non-formal manifold with $b_1(M) = 1$ and $b_2(M) = 2$.*

Proof. ad (i): Let $N = U_3(\mathbb{R}) \times \mathbb{R}^2$ denote the nilradical of G . We can identify N with \mathbb{R}^5 as a manifold and the multiplication given by

$$(a, b, c, r, s) \cdot (x, y, z, v, w) = (a + x + bz, b + y, c + z, r + v, s + w),$$

i.e. $[N, N] = \{(x, 0, 0, 0, 0) \mid x \in \mathbb{R}\} \cong \mathbb{R}$ and $\overline{N} := N/[N, N] \cong \mathbb{R}^4$.

By definition of G , we have $G = \mathbb{R} \rtimes_{\mu} N$, where

$$\forall_{t \in \mathbb{R}} \mu(t) = \exp^N \circ \exp^{A(n)}(t \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}) \circ \log^N, \quad (3.9)$$

and since $\exp^{\mathbb{R}^4} = \text{id}$, the induced maps $\bar{\mu}(t): \bar{N} \rightarrow \bar{N}$ are given by

$$\begin{aligned} \bar{\mu}(t)((y, z, v, w)) &= \exp^{\text{GL}(4, \mathbb{R})}(t \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}) \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ -te^{-t} & 0 & e^{-t} & 0 \\ 0 & -te^t & 0 & e^t \end{pmatrix} \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix}. \end{aligned}$$

One calculates that $\tilde{\mu}: \mathbb{R} \rightarrow \text{A}(N)$ given by

$$\forall t \in \mathbb{R} \quad \forall (x, y, z, v, w) \in N \quad \tilde{\mu}(t)((x, y, z, v, w)) = (x, \bar{\mu}(t)((y, z, v, w))) \quad (3.10)$$

is a one-parameter group, and since the derivations of (3.9) and (3.10) in zero are equal, we have $\mu \equiv \tilde{\mu}$.

Let $t_1 = \ln(\frac{3+\sqrt{5}}{2})$, then $\bar{\mu}(t_1)$ is conjugate to $A := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. The

transformation matrix $T \in \text{GL}(4, \mathbb{R})$ with $TAT^{-1} = \bar{\mu}(t_1)$ is

$$T = \begin{pmatrix} 1 & -\frac{2(2+\sqrt{5})}{3+\sqrt{5}} & 0 & 0 \\ 1 & \frac{1+\sqrt{5}}{3+\sqrt{5}} & 0 & 0 \\ 0 & 0 & \ln(\frac{2}{3+\sqrt{5}}) & \frac{2(2+\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} \\ 0 & 0 & \ln(\frac{2}{3+\sqrt{5}}) & -\frac{(1+\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} \end{pmatrix}.$$

Denote by $\{b_1, \dots, b_4\}$ the basis of \mathbb{R}^4 for which $\bar{\mu}(t_1)$ is represented by A , i.e. b_i is the i -th column of T . One calculates

$$\begin{aligned} b_{11}b_{22} - b_{12}b_{21} &= \sqrt{5}, \\ b_{i1}b_{j2} - b_{i2}b_{j1} &= 0 \quad \text{for } i < j, (i, j) \neq (1, 2). \end{aligned}$$

This implies that we have for $\gamma_0 := (\sqrt{5}, 0_{\mathbb{R}^4})$, $\gamma_i := (b_{i0}, b_i) \in N$ with arbitrary $b_{i0} \in \mathbb{R}$, $i = 1, \dots, 4$,

$$[\gamma_1, \gamma_2] = \gamma_0, \quad [\gamma_1, \gamma_3] = [\gamma_1, \gamma_4] = [\gamma_2, \gamma_3] = [\gamma_2, \gamma_4] = [\gamma_3, \gamma_4] = e_N.$$

We can choose the b_{i0} such that the following equations hold:

$$\begin{aligned} \mu(t_1)(\gamma_0) &= \gamma_0, \\ \mu(t_1)(\gamma_1) &= \gamma_1^2 \quad \gamma_2 \quad \gamma_3^2 \quad \gamma_4, \\ \mu(t_1)(\gamma_2) &= \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \gamma_4, \\ \mu(t_1)(\gamma_3) &= \gamma_3^2 \quad \gamma_4, \\ \mu(t_1)(\gamma_4) &= \gamma_3 \quad \gamma_4. \end{aligned} \quad (3.11)$$

Note that (3.11) leads to the equation $(\text{id} - \tau A) \begin{pmatrix} b_{10} \\ b_{20} \\ b_{30} \\ b_{40} \end{pmatrix} = \begin{pmatrix} 1 + \frac{2(1+\sqrt{5})}{3+\sqrt{5}} \\ \frac{1+\sqrt{5}}{3+\sqrt{5}} \\ 0 \\ 0 \end{pmatrix}$

which has the (unique) solution $b_{10} = -\frac{1+\sqrt{5}}{3+\sqrt{5}}$, $b_{20} = -\frac{11+5\sqrt{5}}{7+3\sqrt{5}}$ and $b_{30} = b_{40} = 0$.

We claim that $t_1\mathbb{Z} \times_{\mu} \langle \exp^N(\text{Span}_{\mathbb{Z}} \log^N(\{\gamma_0, \dots, \gamma_4\})) \rangle$ defines a lattice in G :

It suffices to show that $\langle \exp^N(\text{Span}_{\mathbb{Z}} \log^N(\{\gamma_0, \dots, \gamma_4\})) \rangle$ defines a lattice in N , so let us prove this assertion. There exist uniquely $Y_0, \dots, Y_4 \in \mathfrak{n}$ with $\exp^N(Y_i) = \gamma_i$ for $i \in \{0, \dots, 4\}$. If we prove that $\mathfrak{Y} := \{Y_0, \dots, Y_4\}$ is a basis of \mathfrak{n} with rational structure constants, then Theorem 3.1.1 (i) implies that $\langle \exp^N(\text{Span}_{\mathbb{Z}} \mathfrak{Y}) \rangle$ is a lattice in N .

We identify \mathfrak{n} with \mathbb{R}^5 and brackets given by the Campbell-Hausdorff formula, see e.g. [76, Chapter 2.15]. Since \mathfrak{n} is 2-step nilpotent (and \exp^N is a diffeomorphism), the formula yields for all $V, W \in \mathfrak{n}$

$$\log^N(\exp^N(V) \exp^N(W)) = V + W + \frac{1}{2}[V, W].$$

Since $U_3(\mathbb{R})$ can be considered as a group of matrices, one can easily calculate its exponential map. Then, its knowledge implies that the exponential map resp. the logarithm of N is given by

$$\begin{aligned} \exp^N((x, y, z, v, w)) &= (x + \frac{1}{2}yz, y, z, v, w), \\ \log^N((x, y, z, v, w)) &= (x - \frac{1}{2}yz, y, z, v, w), \end{aligned}$$

and we obtain $Y_0 = (\sqrt{5}, 0_{\mathbb{R}^4})$, $Y_1 = (b_{10} - \frac{1}{2}, b_1)$, $Y_2 = (b_{20} + \frac{(2+\sqrt{5})(1+\sqrt{5})}{(3+\sqrt{5})^2}, b_2)$, $Y_3 = (0, b_3)$, $Y_4 = (0, b_4)$, $[Y_1, Y_2] = Y_0$. The other brackets vanish.

ad (ii): Let Γ be an arbitrary lattice in G . By complete solvability and Theorem 3.2.11 (ii), we get the minimal model of $M = G/\Gamma$ as the minimal model \mathcal{M} of the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta)$. The latter has the closed generator x_6 and the non-closed generators satisfy

$$\delta x_1 = -x_{23}, \delta x_2 = -x_{26}, \delta x_3 = x_{36}, \delta x_4 = -x_{26} - x_{46}, \delta x_5 = -x_{36} + x_{56},$$

which implies $b_1(M) = 1$.

One computes the differential of the non-exact generators of degree two in the Chevalley-Eilenberg complex as

$$\begin{aligned} \delta x_{12} &= x_{126}, & \delta x_{13} &= -x_{136}, & \delta x_{14} &= x_{126} + x_{146} - x_{234}, \\ \delta x_{15} &= x_{136} - x_{156} - x_{235}, & \delta x_{16} &= -x_{236}, & \delta x_{24} &= 2x_{246}, \\ \delta x_{25} &= x_{236}, & \delta x_{34} &= -x_{236}, & \delta x_{35} &= -2x_{356}, \\ \delta x_{45} &= x_{256} - x_{346}, \end{aligned}$$

i.e. $b_2(M) = 2$.

The minimal model $\rho: (\wedge V, d) \rightarrow (\wedge \mathfrak{g}^*, \delta)$ must contain three closed generators y, z_1, z_2 which map to $x_6, x_{16} + x_{25}$ and $x_{16} - x_{34}$. $\rho(yz_1) = x_{256}$ and $\rho(yz_2) = -x_{346}$ are closed and not exact. But in the generation of y, z_1 and z_2 is one (and up to a scalar only one) element that maps onto an exact form, namely $\rho(y(z_1 + z_2)) = \delta x_{45}$. The minimal model's construction in the proof of Theorem 1.1.2 implies that there is another generator u of degree two such that $\rho(u) = x_{45}$ and $du = y(z_1 + z_2)$. Since $\rho(yu) = x_{456}$ is closed and non-exact, there are no further generators of degree less than or equal to two in V . But this implies for each closed element c of degree two that $y(u + c)$ is closed and non-exact in \mathcal{M} . Using the notation of Theorem 1.1.5, we have $u \in N^2, y \in V^1$ and \mathcal{M} is not formal.

Finally, the existence of a symplectic form on G/Γ follows from Proposition 3.8.3.1. \square

Proposition 3.8.3.3.

- (i) *Each quotient of the Lie group $G_{6.38}^0$ by a lattice is symplectic. $G_{6.38}^0$ contains a lattice Γ with $b_1(G_{6.38}^0/\Gamma) = 1$.*
- (ii) *If the Lie group $G_{6.38}^0$ contains a lattice Γ such that $M := G_{6.38}^0/\Gamma$ satisfies $b_1(M) = 1$ and $b_2(M) = 2$, then M is a symplectic and non-formal manifold.*

Proof. The proof is similar to that of the last theorem. Therefore, we just give a sketch of the proof.

ad (i): The existence of a symplectic form on each quotient of $G := G_{6.38}^0$ by a lattice follows from Proposition 3.8.3.1.

The nilradical N of G is the same as in the proof of Theorem 3.8.3.2, so we have $[N, N] = \mathbb{R}$ and $\overline{N} = N/[N, N] = \mathbb{R}^4$. If $\overline{\mu}(t): \overline{N} \rightarrow \overline{N}$ is defined by

$$\begin{aligned} \overline{\mu}(t)((y, z, v, w)) &= \exp^{\text{GL}(4, \mathbb{R})}(t \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}) \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix} \\ &= \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 \\ -t \cos(t) & -t \sin(t) & \cos(t) & \sin(t) \\ t \sin(t) & -t \cos(t) & -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix}, \end{aligned}$$

one calculates that $\mu: \mathbb{R} \rightarrow \text{A}(N)$ given by

$$\begin{aligned} \mu(t)((x, y, z, v, w)) &= (x - \sin^2(t)yz + \frac{\sin(t) \cos(t)}{2}(z^2 - y^2) + t \frac{\sqrt{3}}{8}(y - z), \\ &\quad \overline{\mu}(t)((y, z, v, w))) \end{aligned}$$

is a one-parameter group with $d_e(\mu(t)) = \exp^{A(n)}(t \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}}_{= \text{ad}(X_6)|_{\mathfrak{n}}})$, i.e.

$G = \mathbb{R} \rtimes_{\mu} N$. (Here X_6 is chosen as in the last line of Table A.14 on page 110.) For $t_1 := \frac{\pi}{3}$ we have

$$\mu(t_1)((x, y, z, v, w)) = \left(x - \frac{3}{4}yz + \frac{\sqrt{3}}{8}(z^2 - y^2) + \frac{\pi}{8\sqrt{3}}(y - z), \bar{\mu}(t)((y, z, v, w))\right),$$

and in order to construct a lattice in G , it is enough to construct a lattice in N

that is preserved by $\mu(t_1)$. $\bar{\mu}(t_1)$ is conjugate to $A := \begin{pmatrix} -1 & -3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -2 & -3 & -1 & -3 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

and the transformation matrix $T \in \text{GL}(4, \mathbb{R})$ with $TAT^{-1} = \bar{\mu}(t_1)$ is

$$T = \begin{pmatrix} \frac{\sqrt{3}}{\pi} & 0 & 0 & 0 \\ -\frac{3}{\pi} & -\frac{6}{\pi} & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} & -\sqrt{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Denote by $\{b_1, \dots, b_4\}$ the basis of \mathbb{R}^4 for which $\bar{\mu}(t_1)$ is represented by A , i.e. b_i is the i -th column of T . One calculates

$$\begin{aligned} b_{11}b_{22} - b_{12}b_{21} &= \frac{-6\sqrt{3}}{\pi^2}, \\ b_{i1}b_{j2} - b_{i2}b_{j1} &= 0 \quad \text{for } i < j, (i, j) \neq (1, 2). \end{aligned}$$

This implies that we have for $\gamma_0 := (b_{11}b_{22} - b_{12}b_{21}, 0_{\mathbb{R}^4})$, $\gamma_i := (b_{i0}, b_i) \in N$ with arbitrary $b_{i0} \in \mathbb{R}$, $i = 1, \dots, 4$,

$$[\gamma_1, \gamma_2] = \gamma_0, \quad [\gamma_1, \gamma_3] = [\gamma_1, \gamma_4] = [\gamma_2, \gamma_3] = [\gamma_2, \gamma_4] = [\gamma_3, \gamma_4] = e_N.$$

If we set $b_{10} = \frac{1488\sqrt{3}+72\sqrt{3}\pi-19\sqrt{3}\pi^2+4\pi^3}{128\pi^2}$, $b_{20} = \frac{2736\sqrt{3}+216\sqrt{3}\pi-25\sqrt{3}\pi^2+12\pi^3}{128\pi^2}$ and $b_{30} = b_{40} = 0$, we obtain

$$\begin{aligned} \mu(t_1)(\gamma_0) &= \gamma_0, \\ \mu(t_1)(\gamma_1) &= \gamma_1^{-1} \gamma_2 \gamma_3^{-2} \gamma_4, \\ \mu(t_1)(\gamma_2) &= \gamma_1^{-3} \gamma_2^2 \gamma_3^{-3} \gamma_4, \\ \mu(t_1)(\gamma_3) &= \gamma_3^{-1} \gamma_4, \\ \mu(t_1)(\gamma_4) &= \gamma_3^{-3} \gamma_4^2. \end{aligned}$$

Then $\langle \exp^N(\text{Span}_{\mathbb{Z}} \log^N(\{\gamma_0, \dots, \gamma_4\})) \rangle$ is a lattice in N . This can be seen by a similar computation as in the proof of the last theorem. Finally, one checks that the abelianisation of this lattice is isomorphic to \mathbb{Z} , hence the corresponding solvmanifold has $b_1 = 1$.

ad (ii): Let Γ be a lattice in G such that $b_1(G/\Gamma) = 1$ and $b_2(G/\Gamma) = 2$.

The Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta)$ has the closed generator x_6 and δ is given on the non-closed generators by

$$\delta x_1 = -x_{23}, \delta x_2 = x_{36}, \delta x_3 = -x_{26}, \delta x_4 = -x_{26} + x_{56}, \delta x_5 = -x_{36} - x_{46},$$

which implies $H^1(\bigwedge \mathfrak{g}^*, \delta) = \langle [x_6] \rangle$.

One computes the differential of the non-exact generators of degree two in the Chevalley-Eilenberg complex as

$$\begin{aligned} \delta x_{12} &= -x_{136}, & \delta x_{13} &= x_{126}, \\ \delta x_{14} &= x_{126} - x_{156} - x_{234}, & \delta x_{15} &= x_{136} + x_{146} - x_{235}, \\ \delta x_{16} &= -x_{236}, & \delta x_{24} &= -x_{256} - x_{346}, \\ \delta x_{25} &= x_{236} + x_{246} - x_{356}, & \delta x_{34} &= -x_{236} + x_{246} - x_{356}, \\ \delta x_{35} &= x_{256} + x_{346}, & \delta x_{45} &= x_{256} - x_{346}, \end{aligned}$$

i.e. $H^2(\bigwedge \mathfrak{g}^*, \delta) = \langle [x_{16} + \frac{1}{2}x_{25} - \frac{1}{2}x_{34}], [x_{24} + x_{35}] \rangle$.

This implies that G/Γ and $(\bigwedge \mathfrak{g}^*, \delta)$ have the same Betti numbers and therefore, by Theorem 3.2.11, they share their minimal model.

The minimal model $\rho: (\bigwedge V, d) \rightarrow (\bigwedge \mathfrak{g}^*, \delta)$ must contain three closed generators y, z_1, z_2 which map to $x_6, x_{16} + \frac{1}{2}x_{25} - \frac{1}{2}x_{34}$ and $x_{24} + x_{35}$. $\rho(yz_2) = x_{246} + x_{356}$ is closed and not exact. But $\rho(yz_1) = \frac{1}{2}(x_{256} - x_{346}) = \frac{1}{2}\delta x_{45}$ is exact. Hence the minimal model's construction in the proof of Theorem 1.1.2 implies that there is another generator u of degree two such that $\rho(u) = \frac{1}{2}x_{45}$ and $du = yz_1$. Since $\rho(yu) = \frac{1}{2}x_{456}$ is closed and non-exact, there are no further generators of degree less than or equal to two in V . Using the notation of Theorem 1.1.5, we have $u \in N^2, y \in V^1, (u + c)y$ is closed and not exact for each $c \in C^2$ and $(\bigwedge V, d)$ is not formal.

Finally, the existence of a symplectic form on G/Γ follows from Proposition 3.8.3.1. \square

Theorem 3.8.3.4.

- (i) *The completely solvable Lie group $G := G_{6,78}$ possesses a lattice.*
- (ii) *For each lattice the corresponding quotient is a symplectic and formal manifold with $b_1 = b_2 = 1$.*

Proof. ad (i): By definition, we have $G = \mathbb{R} \times_{\mu} N$ with $N = G_{5,3}$ and $\mu(t) = \exp^N \circ \exp^{A(n)}(t \text{ad}(X_6)|_{\mathfrak{n}}) \circ \log^N$, where $\{X_1, \dots, X_6\}$ denotes a basis of

\mathfrak{g} as in the second row of Table A.20. Note that $\{X_1, \dots, X_5\}$ is a basis for the nilradical \mathfrak{n} . One computes

$$\mu(t)_* := d_e(\mu(t)) = \exp^{A(\mathfrak{n})}(t \operatorname{ad}(X_6)|_{\mathfrak{n}}) = \begin{pmatrix} e^t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-t} & -te^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 0 & e^t \end{pmatrix}. \quad (3.12)$$

Using

$$\mathfrak{n} = \langle X_5 \rangle \ltimes_{\operatorname{ad}(X_5)} (\langle X_1 \rangle \oplus \langle X_2, X_3, X_4 \mid [X_2, X_4] = X_3 \rangle)$$

with $\operatorname{ad}(X_5)(X_2) = -X_1$, $\operatorname{ad}(X_5)(X_4) = -X_2$, $\operatorname{ad}(X_5)(X_1) = \operatorname{ad}(X_5)(X_3) = 0$ and

$$\langle X_2, X_3, X_4 \mid [X_2, X_4] = X_3 \rangle \cong \mathfrak{g}_{3.1},$$

we can determine the Lie group N .

As a smooth manifold N equals \mathbb{R}^5 , and the multiplication is given by

$$\begin{aligned} & (a, b, c, r, s) \cdot (x, y, z, v, w) \\ &= \left(a + x + bw + \frac{rw^2}{2}, b + y + rw, c + z + bv + \frac{r^2w}{2} + rvw, r + v, s + w \right). \end{aligned}$$

Now, Theorem 3.3.2 enables us to compute the exponential map of N as

$$\begin{aligned} & \exp^N(xX_1 + yX_2 + zX_3 + vX_4 + wX_5) \\ &= \left(x + \frac{yw}{2} + \frac{vw^2}{6}, y + \frac{vw}{2}, z + \frac{yv}{2} + \frac{v^2w}{3}, v, w \right), \end{aligned}$$

and therefore, we also obtain the logarithm of N

$$\begin{aligned} & \log^N((x, y, z, v, w)) \\ &= \left(x - \frac{yw}{2} + \frac{vw^2}{12} \right) X_1 + \left(y - \frac{vw}{2} \right) X_2 + \left(z - \frac{yv}{2} - \frac{v^2w}{12} \right) X_3 + vX_4 + wX_5. \end{aligned}$$

Finally, a short computation shows that (3.12) implies

$$\mu(t)((x, y, z, v, w)) = (e^t x, y, e^{-t}(z - tw), e^{-t}v, e^t w).$$

Let $t_1 := \ln\left(\frac{3+\sqrt{5}}{2}\right)$, $b_0 := -\frac{2t_1}{1+\sqrt{5}}$ and consider for $t \in \mathbb{R}$ the automorphisms $I(t): N \rightarrow N$ given by

$$\begin{aligned} & I(t)((x, y, z, v, w)) \\ &= (0, tb_0, 0, 0, 0)(x, y, z, v, w)(0, tb_0, 0, 0, 0)^{-1} = (x + tb_0 w, y, z + tb_0 v, v, w), \end{aligned}$$

and $\nu(t) := \mu(t) \circ I(t): N \rightarrow N$. It is easy to see that $\nu: \mathbb{R} \rightarrow A(N)$ is a one-parameter group in N .

We shall show that there exists a lattice Γ_N in N preserved by $\nu(t_1)$, and this then implies the existence of a lattice in $G_{6,78}$, namely $t_1\mathbb{Z} \times_{\nu} \Gamma_N$.

For the remainder of the proof, we identify $\mathfrak{n} \equiv \mathbb{R}^5$ with respect to the basis $\{X_1, \dots, X_5\}$ of \mathfrak{n} . Under this identification, consider the basis $\{Y_1, \dots, Y_5\}$ of \mathfrak{n} , Y_i being the i -th column of $T = (T_{ij}) \in \text{GL}(5, \mathbb{R})$, where T has the following entries:

$$\begin{aligned}
T_{11} &= \frac{10(161 + 72\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})^2}{1165 + 521\sqrt{5}}, \\
T_{12} &= 0, \\
T_{13} &= \frac{5(2 + \sqrt{5})(161 + 72\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})^2}{1525 + 682\sqrt{5}}, \\
T_{14} &= \frac{328380 + 146856\sqrt{5} - (159975 + 71543\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})^2}{202950 + 90762\sqrt{5}}, \\
T_{15} &= 1, \\
T_{21} &= 0, \\
T_{22} &= -\frac{(5 + 3\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})}{3 + \sqrt{5}}, \\
T_{23} &= 0, \\
T_{24} &= -\frac{(158114965 + 70711162\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})}{141422324 + 63245986\sqrt{5}}, \\
T_{25} &= \frac{5(3940598 + 1762585\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})}{17622890 + 7881196\sqrt{5}}, \\
T_{31} &= \frac{1}{2}(5 + \sqrt{5}) \ln(\frac{3 + \sqrt{5}}{2}), \\
T_{32} &= 0, \\
T_{33} &= T_{22}, \\
T_{34} &= 1, \\
T_{35} &= -\frac{597 + 267\sqrt{5} + (3808 + 1703\sqrt{5}) \ln(\frac{3+\sqrt{5}}{2})}{369 + 165\sqrt{5}}, \\
T_{41} &= 0, \\
T_{42} &= 0, \\
T_{43} &= 0, \\
T_{44} &= 1,
\end{aligned}$$

$$T_{45} = -\frac{2(2 + \sqrt{5})}{3 + \sqrt{5}},$$

$$T_{51} = 0,$$

$$T_{52} = 0,$$

$$T_{53} = 0,$$

$$T_{54} = \ln\left(\frac{2}{3 + \sqrt{5}}\right),$$

$$T_{55} = -\frac{2 \ln\left(\frac{3+\sqrt{5}}{2}\right)}{1 + \sqrt{5}}.$$

Let $\gamma_i := \exp^N(Y_i)$ for $i \in \{1, \dots, 5\}$ and

$$\begin{aligned} S_1 &= \frac{92880525355200 + 41537433696024\sqrt{5}}{57403321562460 + 25671545829588\sqrt{5}} \\ &\quad - \frac{(3591421616495 + 1606132574069\sqrt{5}) \ln\left(\frac{3+\sqrt{5}}{2}\right)^2}{57403321562460 + 25671545829588\sqrt{5}}, \\ S_2 &= -\frac{(228826127 + 102334155\sqrt{5}) \ln\left(\frac{3+\sqrt{5}}{2}\right)}{141422324 + 63245986\sqrt{5}}, \\ S_3 &= 1 - \frac{(757189543 + 338625458\sqrt{5}) \ln\left(\frac{3+\sqrt{5}}{2}\right)}{848533944 + 379475916\sqrt{5}}, \\ S_4 &= \frac{724734510 + 324111126\sqrt{5} - (325041375 + 145362922\sqrt{5}) \ln\left(\frac{3+\sqrt{5}}{2}\right)^2}{724734510 + 324111126\sqrt{5}}, \\ S_5 &= \frac{(120789085 + 54018521\sqrt{5}) \ln\left(\frac{3+\sqrt{5}}{2}\right)}{74651760 + 33385282\sqrt{5}}, \\ S_6 &= -\frac{466724522940 + 208725552012\sqrt{5}}{24(12018817440 + 5374978561\sqrt{5})} \\ &\quad + \frac{(3393446021605 + 1517595196457\sqrt{5}) \ln\left(\frac{3+\sqrt{5}}{2}\right)}{24(12018817440 + 5374978561\sqrt{5})}. \end{aligned}$$

One computes $\gamma_1 = (T_{11}, 0, T_{31}, 0, 0)$, $\gamma_2 = (0, T_{22}, 0, 0, 0)$, $\gamma_3 = (T_{13}, 0, T_{33}, 0, 0)$, $\gamma_4 = (S_1, S_2, S_3, T_{44}, T_{54})$ and $\gamma_5 = (S_4, S_5, S_6, T_{45}, T_{55})$.

Moreover, if A denotes the matrix $\begin{pmatrix} 1 & 0 & 1 & \frac{13}{6} & \frac{11}{6} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 1 & 0 & 2 & -\frac{5}{6} & -\frac{1}{3} \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$, we can calculate

$TAT^{-1} = \nu(t_1)_* := d_e(\nu(t))$. Since $\nu(t_1) = \exp^N \circ \nu(t_1)_* \circ \log^N$, this yields $\nu(t_1)(\gamma_1) = \gamma_1 \gamma_3$, $\nu(t_1)(\gamma_2) = \gamma_2$, $\nu(t_1)(\gamma_3) = \gamma_1 \gamma_3^2$, $\nu(t_1)(\gamma_4) = \gamma_1^2 \gamma_2^{-2} \gamma_4^2 \gamma_5$ and $\nu(t_1)(\gamma_5) = \gamma_1^2 \gamma_2^{-1} \gamma_4 \gamma_5$.

Therefore, we have shown that $\nu(t_1)$ preserves the subgroup Γ_N of N which is generated by $\gamma_1, \dots, \gamma_5$. In order to complete the proof of (i), it suffices to show that Γ_N is a lattice in N .

Since \mathfrak{n} is 3-step nilpotent, the Baker-Campbell-Hausdorff formula (see e.g. [76, Chapter 2.15]) yields for all $V, W \in \mathfrak{n}$

$$\log^N(\exp^N(V)\exp^N(W)) = V + W + \frac{1}{2}[V, W] + \frac{1}{12}([[[V, W], W] - [[V, W], V]]).$$

Therefore, we obtain by a short calculation $[Y_2, Y_4] = Y_3$, $[Y_2, Y_5] = Y_1$ and $[Y_4, Y_5] = \frac{1}{2}Y_1 + Y_2 + \frac{1}{2}Y_3$, i.e. the basis $\{Y_1, \dots, Y_5\}$ has rational structure constants. Theorem 3.1.1 then implies that Γ_N is a lattice in N .

ad (ii): Let Γ be a lattice in $G := G_{6,78}$. By complete solvability and Theorem 3.2.11 (ii), the minimal model of $M = G/\Gamma$ is the same as the minimal model \mathcal{M} of the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta)$. In view of Theorem 1.1.6, it suffices to prove that the latter is 2-formal. On the non-closed generators of $(\bigwedge \mathfrak{g}^*, \delta)$ the differential is given by

$$\delta x_1 = x_{16} - x_{25}, \delta x_2 = -x_{45}, \delta x_3 = -x_{24} - x_{36} - x_{46}, \delta x_4 = -x_{46}, \delta x_5 = x_{56},$$

i.e. $H^1(\bigwedge \mathfrak{g}^*, \delta) = \langle [x_6] \rangle$. Further, one calculates $H^2(\bigwedge \mathfrak{g}^*, \delta) = \langle [x_{14} + x_{26} + x_{35}] \rangle$. The minimal model $\rho: (\bigwedge V, d) \rightarrow (\bigwedge \mathfrak{g}^*, \delta)$ then must contain two closed generators y, z which map to x_6 and $x_{14} + x_{26} + x_{35}$. Since $\rho(yz) = x_{146} + x_{356}$ is closed and non-exact, there are no other generators of degree two in $(\bigwedge V, d)$, hence up to degree two, all generators are closed. This implies the minimal model's 2-formality.

Moreover, $x_{14} + x_{26} + x_{35}$ defines a symplectic form. \square

Remark. In order to determine a lattice in $G_{6,78}$, the author also found a lattice of the completely solvable Lie group $G_{6,76}^{-1}$. One can show that the corresponding solvmanifold is formal and has first Betti number equal to one. Unfortunately, it is not symplectic by Proposition 3.8.3.1.

Remark. Besides the mentioned groups above, the following non-completely solvable Lie groups $G_{6,i}$ could give rise to a symplectic solvmanifold $G_{6,i}/\Gamma$ with $b_1(G_{6,i}/\Gamma) = 1$:

$$\begin{aligned} i = 8; & & i = 9, b \neq 0; & & i = 10, a \neq 0; \\ i = 11; & & i = 12; & & i = 32, a = \varepsilon = 0 < h; \\ i = 37, a = 0; & & i = 88, \mu_0\nu_0 \neq 0; & & i = 89, \nu_0s \neq 0; \\ i = 90, \nu_0 \neq 0; & & i = 92, \mu_0\nu_0 \neq 0; & & i = 92^*; \\ i = 93, |\nu_0| > \frac{1}{2}. & & & & \end{aligned} \tag{3.13}$$

But then the cohomology class of the symplectic form cannot lie in the image of the inclusion $H^*(\bigwedge \mathfrak{g}_{6,i}^*, \delta) \hookrightarrow H^*(G_{6,i}/\Gamma, d)$ by Proposition 3.8.3.1.

3.8.4 Symplectic solvmanifolds whose first Betti number is greater than one

In this section, we examine which Lie groups G can give rise to a six-dimensional solvmanifold G/Γ with $b_1(G/\Gamma) > 1$. Again, we just consider indecomposable connected and simply-connected solvable Lie groups. The nilradical of such a group has not dimension equal to three, see e.g. [58]. Proposition 3.2.5 then tells us that indecomposable solvable Lie groups have nilradicals of dimension greater than three. Moreover, the nilpotent ones were considered in Section 3.8.1, hence we can assume that G is non-nilpotent, i.e. $\dim \text{Nil}(G) \in \{4, 5\}$. The corresponding Lie algebras are listed in Tables A.10 – A.26.

In Section 3.8.2, we have excluded some groups G since they cannot admit lattices. Clearly, we omit them in the considerations below.

By Theorem 3.2.11(ii), we have in the completely solvable case an isomorphism from Lie algebra cohomology to the solvmanifold's cohomology, i.e. the Lie algebra \mathfrak{g} must satisfy $b_1(\mathfrak{g}) > 1$, too. In the last section, we saw that $\mathfrak{g}_{6.38}^0$ is the only non-completely solvable but cohomologically symplectic Lie algebra with $b_1(\mathfrak{g}) = 1$. Therefore, for each lattice Γ in $G_{6.38}^0$ with $b_1(G_{6.38}^0/\Gamma) > 1$, the quotient is symplectic. We now turn to Lie algebras with $b_1(\mathfrak{g}) > 1$. The possible values of b_1 can be read of in Tables A.27 – A.29.

The remaining algebras to examine are $\mathfrak{g}_{6.i}$ with

$$\begin{array}{lll}
i = 2, a = 0; & i = 3, d = -1; & i = 6, a = -\frac{1}{2}, b = 0; \\
i = 9, b = 0; & i = 10, a = 0; & i = 21, a = 0; \\
i = 23, a = 0; & i = 25, b = 0; & i = 26; \\
i = 29, b = 0; & i = 33, a = 0; & i = 34, a = 0; \\
i = 36, a = 0; & i = 54, l = -1; & i = 63; \\
i = 65, l = 0; & i = 70, p = 0; & i = 83, l = 0; \\
i = 84; & i = 88, \mu_0 = \nu_0 = 0; & i = 89, \nu_0 s = 0; \\
i = 90, \nu_0 = 0; & i = 92, \nu_0 \mu_0 = 0; & i = 93, \nu_0 = 0; \\
i = 102; & i = 105; & i = 107; \\
i = 113; & i = 114; & i = 115; \\
i = 116; & i = 118; & i = 120; \\
i = 125; & i = 129; & i = 135.
\end{array} \tag{3.14}$$

As above, we just consider such Lie algebras that are cohomologically symplectic, although this condition is only in the completely solvable case necessary for the existence of a symplectic form on G/Γ .

Proposition 3.8.4.1. *Let $\mathfrak{g}_{6.i}$ be one of the Lie algebras listed in (3.14).*

Then $\mathfrak{g}_{6.i}$ is cohomologically symplectic if and only if it is contained in the

following list:

$$\begin{aligned}
b_1 = 2 : \quad & i = 3, d = -1; & i = 10, a = 0; & i = 21, a = 0; \\
& i = 36, a = 0; & i = 54, l = -1; \\
& i = 70, p = 0; & i = 118, b = \pm 1. \\
b_1 = 3 : \quad & i = 23, a = 0, \varepsilon \neq 0; & i = 29, b = 0.
\end{aligned}$$

Proof. This is done by a case by case analysis as described in the proof of Proposition 3.8.3.1. We list the symplectic forms for the Lie algebras that are cohomologically symplectic. In the cases with $b_1 = 2$, the symplectic forms are given by

$$\begin{aligned}
i = 3, d = -1 & : \omega = \lambda x_{16} + \mu x_{23} + \nu x_{45}, & \lambda\mu\nu \neq 0, \\
i = 10, a = 0 & : \omega = \lambda x_{16} + \mu x_{23} + \nu x_{45}, & \lambda\mu\nu \neq 0, \\
i = 21, a = 0 & : \omega = \lambda x_{12} + \mu x_{36} + \nu x_{45}, & \lambda\mu\nu \neq 0, \\
i = 36, a = 0 & : \omega = \lambda x_{12} + \mu x_{36} + \nu x_{45}, & \lambda\mu\nu \neq 0, \\
i = 54, l = -1 & : \omega = \lambda(x_{12} + x_{23}) + \mu x_{34} + \nu x_{56}, & \lambda\nu \neq 0, \\
i = 70, p = 0 & : \omega = \lambda(x_{13} + x_{24}) + \mu x_{34} + \nu x_{56}, & \lambda\nu \neq 0, \\
i = 118, b = \pm 1 & : \omega = \lambda(x_{13} \pm x_{24}) + \mu(x_{14} - x_{23}) + \nu x_{56}, & (\lambda^2 + \mu^2)\nu \neq 0.
\end{aligned}$$

In the cases with $b_1 = 3$, we have the symplectic forms

$$\omega = \lambda(x_{12} + \varepsilon x_{35}) + \mu(x_{16} + x_{24}) + \nu(x_{23} - \varepsilon x_{56}) + \rho x_{25} + \sigma x_{46}$$

with $\lambda\mu\nu \neq 0$ for $i = 23$, $a = 0$, $\varepsilon \neq 0$,

$$\omega = \lambda(x_{13} + \varepsilon x_{45}) + \mu(x_{16} + x_{24}) + \nu(x_{23} - \varepsilon x_{56}) + \rho x_{26} + \sigma x_{34}$$

with $\lambda \neq 0, \rho \neq \frac{(\lambda+\varepsilon)\mu\nu}{\lambda}$ for $i = 29$, $b = 0$, $\varepsilon \neq 0$ and

$$\omega = \lambda x_{12} + \mu x_{13} + \nu(x_{16} + x_{24}) + \rho x_{26} + \sigma x_{34} + \tau x_{56}$$

with $\nu(\nu\sigma + \mu\tau) \neq 0$ for $i = 29$, $b = 0$, $\varepsilon \neq 0$. □

Provided there is a lattice in one of the ten Lie groups $G_{6,i}$ in the last proposition whose Lie algebras are cohomologically symplectic, we can ensure that the corresponding solvmanifold is symplectic. In the completely solvable case, i.e. $i \in \{3, 21, 23, 29, 54\}$, we can determine cohomological properties of the potential solvmanifolds.

Proposition 3.8.4.2.

- (i) *There is a lattice in the completely solvable Lie group $G_{6,3}^{0,-1}$.*
- (ii) *For each lattice the corresponding solvmanifold is symplectic, not formal and satisfies $b_1 = 2$ as well as $b_2 = 3$.*

Proof. ad (i) : By definition, we have $G := G_{6.3}^{0,-1} = \mathbb{R} \ltimes_{\mu} N$ with $N = \mathbb{R}^5$ and $\mu(t) = \exp^{\text{GL}(5, \mathbb{R})}(t \text{ad}(X_6)|_{\mathfrak{n}})$, where $X_6 \in \mathfrak{g}_{6.3}^{0,-1}$ is chosen as in Table A.11, i.e.

$$\mu(t) = \begin{pmatrix} 1 & -t & \frac{t^2}{2} & 0 & 0 \\ 0 & 1 & -t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 0 & e^t \end{pmatrix}.$$

Set $t_1 := \ln(\frac{3+\sqrt{5}}{2})$. Then $\mu(t_1)$ is conjugate to $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$. This follows

from (3.4) and the fact that the Jordan form of the upper left block of $\mu(t_1)$ is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Hence G admits a lattice.

ad (ii): By complete solvability and Theorem 3.2.11 (ii), the solvmanifold's minimal model is the same as the minimal model of the Chevalley-Eilenberg complex $(\wedge(x_1, \dots, x_6), \delta)$. In view of Theorem 1.1.6, it suffices to prove that the latter is not 2-formal.

Using the knowledge of the Chevalley-Eilenberg complex, one can compute $\rho: (\wedge(y_1, \dots, y_4, z), d) \rightarrow (\wedge(x_1, \dots, x_6), \delta)$ as the minimal model up to generators of degree two, where

$$\begin{aligned} \rho(y_1) &= x_3, \quad \rho(y_2) = x_6, \quad \rho(y_3) = -x_2, \quad \rho(y_4) = -x_1, \quad \rho(z) = x_4x_5, \\ dy_1 &= dy_2 = 0, \quad dy_3 = y_1y_2, \quad dy_4 = y_2y_3, \quad dz = 0. \end{aligned}$$

This obviously implies the statement about the Betti numbers. Moreover, using the notation of Theorem 1.1.5, we have $C^1 = \langle y_1, y_2 \rangle$, $N^1 = \langle y_3, y_4 \rangle$, and $y_1(y_3+c)$ is closed but not exact for each $c \in C^1$. Hence the minimal model is not 1-formal. \square

Proposition 3.8.4.3.

- (i) *There is a lattice in the completely solvable Lie group $G_{6.21}^0$.*
- (ii) *For each lattice the corresponding solvmanifold is symplectic, not formal and satisfies $b_1 = 2$ as well as $b_2 = 3$.*

Proof. The proof of (ii) is completely analogous to that of (ii) in the last proposition. But this time, the minimal model is given by

$$\begin{aligned} \rho(y_1) &= x_2, \quad \rho(y_2) = x_6, \quad \rho(y_3) = -x_3, \quad \rho(y_4) = x_1, \quad \rho(z) = x_4x_5, \\ dy_1 &= dy_2 = 0, \quad dy_3 = y_1y_2, \quad dy_4 = y_1y_3, \quad dz = 0, \end{aligned}$$

and $y_1(y_4 + c)$ is closed but non-exact for each closed c of degree one.

ad (i): In order to prove the existence of a lattice, we use the same argumentation as in the proof of Theorem 3.8.3.2 (i). (Note that $G_{6.15}^{-1}$ and $G := G_{6.21}^0$ share their nilradical N .) But of course, we now have a different initial data:

$$G = \mathbb{R} \times_{\mu} N \text{ with } \mu(t) = \exp^N \circ \exp^{A(n)}(t \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}) \circ \log^N \text{ and}$$

$$\begin{aligned} \bar{\mu}(t)((y, z, v, w)) &= \exp^{\text{GL}(4, \mathbb{R})}(t \begin{pmatrix} 0 & 0 & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & t \end{pmatrix}) \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix}. \end{aligned}$$

Arguing analogous as in (3.10), one obtains

$$\mu(t)((x, y, z, v, w)) = (x - \frac{t}{2}y^2, \bar{\mu}(t)((y, z, v, w))).$$

$$\text{Let } t_1 = \ln(\frac{3+\sqrt{5}}{2}), A := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & -\frac{1}{t_1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{18+8\sqrt{5}}{7+3\sqrt{5}} & 1 \\ 0 & 0 & \frac{2}{3+\sqrt{5}} & 1 \end{pmatrix}.$$

Then we have $TAT^{-1} = \bar{\mu}(t_1)$. Denote the i -th column of T by b_i . Analogous calculations as in loc. cit. imply the existence of a lattice generated by $\gamma_0 := (\frac{1}{t_1}, 0_{\mathbb{R}^4})$ and $\gamma_i := (b_{i0}, b_i)$, $i \in \{1, \dots, 4\}$, where $b_{20} \in \mathbb{R}$ arbitrary and $b_{10} = -\frac{1}{2t_1}$ as well as $b_{30} = b_{40} = 0$. \square

Proposition 3.8.4.4.

- (i) Let $\varepsilon = \pm 1$. There is a lattice in the completely solvable Lie group $G_{6.23}^{0,0,\varepsilon}$.
- (ii) If there is a lattice in $G_{6.23}^{0,0,\varepsilon}$, $\varepsilon \neq 0$, then the corresponding solvmanifold is symplectic, non-formal and satisfies $b_1 = 3$ as well as $b_2 = 5$.

Proof. ad (i): $G_{6.23}^{0,0,\varepsilon}$ has the same nilradical N as $G_{6.15}^{-1}$ and the latter is described at the beginning of the proof of Theorem 3.8.3.2.

By definition, $G_{6.23}^{0,0,\varepsilon} = \mathbb{R} \times_{\mu} N$ with

$$\mu(t) = \exp^N \circ \exp^{A(n)}(t \begin{pmatrix} 0 & 0 & 0 & 0 & -\varepsilon \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}) \circ \log^N.$$

The functions \exp^N, \log^N also can be found in the proof of Theorem 3.8.3.2. Using their knowledge, we calculate

$$\mu(t)((x, y, z, v, w)) = \left(x - \frac{t}{2}y^2 - t\varepsilon, y, z - ty, \frac{t^2}{2}y - tz + v, w\right).$$

If $\varepsilon = \pm 1$, then the map $\mu(2)$ preserves the lattice

$$\{(x, y, z, v, w) \in N \mid x, y, z, v, w \in \mathbb{Z}\} \subset N.$$

Therefore, $G_{6.23}^{0,0,\varepsilon}$ admits a lattice.

ad (ii): By completely solvability, the Betti numbers of the Chevalley-Eilenberg complex coincide with the solvmanifold's Betti numbers. A short calculation yields the first Betti numbers of the former as $b_1 = 3$ and $b_2 = 5$.

The knowledge of the Chevalley-Eilenberg complex $(\wedge(x_1, \dots, x_6), \delta)$ enables us to compute the first stage of the minimal model as above. It is given by $\rho: (\wedge(y_1, \dots, y_6), d) \rightarrow (\wedge(x_1, \dots, x_6), \delta)$ with

$$\begin{aligned} \rho(y_1) &= x_2, \quad \rho(y_2) = x_5, \quad \rho(y_3) = x_6, \quad \rho(y_4) = -x_3, \quad \rho(y_5) = x_1, \quad \rho(y_6) = -x_4, \\ dy_1 &= dy_2 = dy_3 = 0, \quad dy_4 = y_1y_3, \quad dy_5 = y_1y_4 - \varepsilon y_2y_3, \quad dy_6 = y_3y_4. \end{aligned}$$

Since $y_3(y_6 + c)$ is closed and non-exact for each closed c of degree one, the minimal model is not 1-formal. \square

Proposition 3.8.4.5.

(i) Let $\varepsilon \in \{0, \pm 1\}$. There is a lattice in the completely solvable Lie group $G_{6.29}^{0,0,\varepsilon}$.

(ii) If there is a lattice in $G_{6.29}^{0,0,\varepsilon}$, $\varepsilon \in \mathbb{R}$, then the corresponding solvmanifold is symplectic, non-formal and has $b_1 = 3$ as well as $b_2 = \begin{cases} 5, & \text{if } \varepsilon \neq 0 \\ 6, & \text{if } \varepsilon = 0 \end{cases}$.

Proof. The argumentation is analogous to the last proof, but this time we have

$$\mu(t)((x, y, z, v, w)) = \left(x - \frac{\varepsilon}{6}t^3z + \frac{\varepsilon}{2}t^2v - \varepsilon tw, y, z, -tz + v, \frac{1}{2}t^2z - tv + w\right).$$

(Note that there is no misprint. The maps $\exp^N \circ \exp^{A(n)}(t \operatorname{ad}(X_6)) \circ \log^N$ and $\exp^{A(n)}(t \operatorname{ad}(X_6))$ have the same form.) For $\varepsilon \in \{0, \pm 1\}$, $\mu(6)$ preserves the integer lattice mentioned in the last proof. This implies (i).

In order to prove (ii), we consider the minimal model. Up to generators of degree one, it is given by

$$\begin{aligned} \rho(y_1) &= x_2, \quad \rho(y_2) = x_3, \quad \rho(y_3) = x_6, \quad \rho(y_4) = -x_4, \quad \rho(y_5) = -x_5, \quad \rho(y_6) = -x_1, \\ dy_1 &= dy_2 = dy_3 = 0, \quad dy_4 = y_2y_3, \quad dy_5 = y_3y_4, \quad dy_6 = y_1y_2 + \varepsilon y_3y_5, \end{aligned}$$

if $\varepsilon \neq 0$, and

$$\begin{aligned} \rho(y_1) = x_2, \rho(y_2) = x_3, \rho(y_3) = x_6, \rho(y_4) = -x_1, \rho(x_5) = -x_4, \rho(y_6) = -x_5, \\ dy_1 = dy_2 = dy_3 = 0, dy_4 = y_1y_2, dy_5 = y_2y_3, dy_6 = y_3y_5, \end{aligned}$$

if $\varepsilon = 0$. In both cases $y_2(y_4 + c)$ is closed and non-exact for all closed c of degree one. \square

The following result is due to Fernández, de León and Saralegui. Its proof can be found in [26, Section 3]. Note that the cohomological results are independent of the choice of the lattice, since the Lie group in the proposition is completely solvable.

Proposition 3.8.4.6. *The completely solvable Lie group $G_{6,54}^{0,-1}$ admits a lattice. For each such, the corresponding solvmanifold is symplectic, non-formal and satisfies $b_1 = 2$ as well as $b_2 = 5$.* \square

Summing up the results concerning completely solvable Lie groups that admit symplectic quotients, we obtain:

Theorem 3.8.4.7. *All six-dimensional symplectic solvmanifolds that can be written as quotient of a non-nilpotent completely solvable indecomposable Lie group are contained in one of the last five propositions, Theorem 3.8.3.2 or Theorem 3.8.3.4.* \square

To end this section, we consider the four cohomologically symplectic Lie algebras $\mathfrak{g}_{6,i}$ of Proposition 3.8.4.1 that are not completely solvable, this means $i = 10 \wedge a = 0$, $i = 36 \wedge a = 0$, $i = 70 \wedge p = 0$ or $i = 118 \wedge b = \pm 1$. Clearly, the existence of a lattice implies that the corresponding solvmanifold is symplectic. But in order to make a statement about cohomological properties, one needs an assumption about the first two Betti numbers to ensure the knowledge of the cohomology algebra.

Proposition 3.8.4.8.

- (i) *Each quotient of the Lie group $G := G_{6,10}^{0,0}$ by a lattice is symplectic and G admits a lattice Γ with $b_1(G/\Gamma) = 2$.*
- (ii) *If there is a lattice in G such that the corresponding solvmanifold satisfies $b_1 = 2$ and $b_2 = 3$, then it is symplectic and not formal.*

Proof. We have $G = \mathbb{R} \times_{\mu} N$ with $N = \mathbb{R}^5$, $\mu(t) = \exp^{\text{GL}(5,\mathbb{R})}(t \text{ad}(X_6)|_{\mathfrak{n}})$ and $X_6 \in \mathfrak{g}_{6,10}^{0,0}$ chosen as in Table A.11, i.e.

$$\mu(t) = \begin{pmatrix} 1 & -t & \frac{t^2}{2} & 0 & 0 \\ 0 & 1 & -t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & 0 & \sin(t) & \cos(t) \end{pmatrix}.$$

$\mu(\pi)$ is conjugate to $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$. This follows from the fact that the

Jordan form of the upper left block of $\mu(\pi)$ is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Hence G admits a

lattice Γ .

A short calculation yields that the abelianisation of this lattice is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$, i.e. $b_1(G/\Gamma) = 2$.

Using the assumptions of (ii), one calculates the minimal model up to generators of degree one as

$$\begin{aligned} \rho(y_1) &= x_3, \quad \rho(y_2) = x_6, \quad \rho(y_3) = -x_2, \quad \rho(y_4) = -x_1, \\ dy_1 &= dy_2 = 0, \quad dy_3 = y_1y_2, \quad dy_4 = y_2y_3, \end{aligned}$$

and $y_1(y_3 + c)$ is closed but not exact for each closed c of degree one. \square

Proposition 3.8.4.9.

- (i) *Each quotient of the Lie group $G := G_{6,36}^{0,0}$ by a lattice is symplectic and G admits a lattice Γ with $b_1(G/\Gamma) = 2$.*
- (ii) *If there is a lattice in the Lie group G such that the corresponding solvmanifold satisfies $b_1 = 2$ and $b_2 = 3$, then it is symplectic and not formal.*

Proof. The proof of (ii) is analogous to the last one. Up to generators of degree one, the minimal model is given by

$$\begin{aligned} \rho(y_1) &= x_2, \quad \rho(y_2) = x_6, \quad \rho(y_3) = -x_3, \quad \rho(y_4) = x_1, \\ dy_1 &= dy_2 = 0, \quad dy_3 = y_1y_2, \quad dy_4 = y_1y_3, \end{aligned}$$

and $y_1(y_4 + c)$ is closed but not exact for each closed c of degree one.

ad (i): Using another initial data, we argue as in the proof of Proposition 3.8.4.3. We now have $\mu(t)((x, y, z, v, w)) = (x - \frac{t}{2}y^2, \bar{\mu}(t)((y, z, v, w)))$ with

$$\begin{aligned} \bar{\mu}(t)((y, z, v, w)) &= \exp^{\text{GL}(4, \mathbb{R})}(t) \begin{pmatrix} 0 & 0 & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & -t & 0 \end{pmatrix} \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ 0 & 0 & \cos(t) & \sin(t) \\ 0 & 0 & -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix}. \end{aligned}$$

Let $t_1 = \pi$, $A := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ and $T := \begin{pmatrix} 0 & -\frac{1}{t_1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then we

have $TAT^{-1} = \bar{\mu}(t_1)$. Denote the i -th column of T by b_i . Analogous calculations as in loc. cit. lead to a lattice generated by $\gamma_0 := (\frac{1}{t_1}, 0_{\mathbb{R}^4})$ and $\gamma_i := (b_{i0}, b_i)$ for $i \in \{1, \dots, 4\}$, where $b_{20} \in \mathbb{R}$ arbitrary and $b_{10} = -\frac{1}{2t_1}$, $b_{30} = b_{40} = 0$.

Obviously, this lattice is represented by

$$\langle \tau, \gamma_0, \dots, \gamma_4 \mid [\tau, \gamma_1] = 1, [\tau, \gamma_2] = \gamma_1, [\tau, \gamma_3] = \gamma_3^{-2}, [\tau, \gamma_4] = \gamma_4^{-2}, [\gamma_1, \gamma_2] = \gamma_0 \rangle$$

and its abelianisation is $\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$, i.e. the solvmanifold's first Betti number equals two. \square

Proposition 3.8.4.10.

(i) *Each quotient of the Lie group $G := G_{6,70}^{0,0}$ by a lattice is symplectic and G admits a lattice Γ with $b_1(G/\Gamma) = 2$.*

(ii) *If there is a lattice Γ in G such that $b_1(G/\Gamma) = 2$ and $b_2(G/\Gamma) = 3$, then G/Γ is formal.*

Proof. ad (i): By definition, we have $G = \mathbb{R} \ltimes_{\mu} N$ with $N = G_{5,1}$ and $\mu(t) = \exp^N \circ \exp^{A(\mathfrak{n})}(t \operatorname{ad}(X_6)|_{\mathfrak{n}}) \circ \log^N$, where $\{X_1, \dots, X_6\}$ denotes a basis of \mathfrak{g} as in the second row of Table A.18. Note that $\{X_1, \dots, X_5\}$ is a basis of the nilradical \mathfrak{n} . One computes

$$\begin{aligned} \mu(t)_* &:= d_e(\mu(t)) = \exp^{A(\mathfrak{n})}(t \operatorname{ad}(X_6)) \\ &= \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 & 0 \\ 0 & 0 & \cos(t) & \sin(t) & 0 \\ 0 & 0 & -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Using $\mathfrak{n} = \langle X_5 \rangle \ltimes_{\operatorname{ad}(X_5)} \langle X_1, \dots, X_4 \rangle$ and $\operatorname{ad}(X_5)(X_1) = \operatorname{ad}(X_5)(X_2) = 0$, $\operatorname{ad}(X_5)(X_3) = -X_1$, $\operatorname{ad}(X_5)(X_4) = -X_2$, we can determine the Lie group N .

As a smooth manifold N equals \mathbb{R}^5 , and the multiplication is given by

$$(a, b, c, r, s) \cdot (x, y, z, v, w) = (a + x + cw, b + y + rw, c + z, r + v, s + w).$$

By Theorem 3.3.2, we can obtain the exponential map of N as

$$\exp^N(xX_1 + yX_2 + zX_3 + vX_4 + wX_5) = \left(x + \frac{zw}{2}, y + \frac{vw}{2}, z, v, w\right),$$

and obviously, this implies

$$\log^N((x, y, z, v, w)) = \left(x - \frac{wz}{2}\right)X_1 + \left(y - \frac{vw}{2}\right)X_2 + zX_3 + vX_4 + wX_5.$$

From $\mu(t) = \exp^N \circ \mu(t)_* \circ \log^N$ we get

$$\begin{aligned} \mu(t)((x, y, z, v, w)) &= (\cos(t)x + \sin(t)y, -\sin(t)x + \cos(t)y, \\ &\quad \cos(t)z + \sin(t)v, -\sin(t)z + \cos(t)v, w) \end{aligned}$$

and $\mu(\pi)$ preserves the lattice $\{(x, y, z, v, w) \in N \mid x, y, z, v, w \in \mathbb{Z}\}$.

The corresponding solvmanifold has $b_1 = 2$ since the abelianisation of this lattice is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}_2^4$.

ad (ii): Up to generators of degree two, the minimal model is given by

$$\begin{aligned} \rho(y_1) &= x_5, \quad \rho(y_2) = x_6, \quad \rho(z_1) = x_{13} + x_{24}, \quad \rho(z_2) = x_{34}, \\ dy_1 &= dy_2 = 0, \quad dz_1 = dz_2 = 0, \end{aligned}$$

hence it is 2-formal. By Theorem 1.1.6, the solvmanifold is formal. \square

Proposition 3.8.4.11.

- (i) $G := G_{6.118}^{0, \pm 1, -1}$ admits a lattice such that the first Betti number of the corresponding solvmanifold equals two (and the second Betti number equals five).
- (ii) If there is a lattice Γ in G such that $b_1(G/\Gamma) = 2$ and $b_2(G/\Gamma) = 3$, then G/Γ is symplectic and formal.

Proof. The construction of the lattices mentioned in (i) can be found in [79]. In loc. cit. $G_{6.118}^{0,1,-1}$ is denoted by G_3 and $G_{6.118}^{0,-1,-1}$ by G_1 , respectively. The Betti numbers of the quotient of $G_{118}^{0,-1,-1}$ are determined explicitly. In the case of $G_{118}^{0,1,-1}$, one can make an analogous computation.

Assume there is a lattice that satisfies the condition of (ii). Up to generators of degree two, the solvmanifold's minimal model is given by

$$\begin{aligned} \rho(y_1) &= x_5, \quad \rho(y_2) = x_6, \quad \rho(z_1) = x_{13} \pm x_{24}, \quad \rho(z_2) = x_{14} \mp x_{23}, \\ dy_1 &= dy_2 = 0, \quad dz_1 = dz_2 = 0, \end{aligned}$$

hence it is 2-formal. Theorem 1.1.6 then implies formality. \square

Remark. $G_{6.118}^{0,-1,-1}$ is the underlying real Lie group of the unique connected and simply-connected complex three-dimensional Lie group that is solvable and not nilpotent. Its compact quotients by lattices are classified in [59, Theorem 1]. They always satisfy $b_1 = 2$ and moreover, for the Hodge number $h^{0,1}$ holds either $h^{0,1} = 1$ or $h^{0,1} = 3$.

Remark. Besides the groups mentioned in this section, the following solvable but not completely solvable Lie groups $G_{6.i}$ could give rise to a symplectic solvmanifold with $b_1 > 1$. But then the cohomology class of the symplectic form cannot lie in the image of the inclusion $H^*(\bigwedge \mathfrak{g}_{6.i}^*, \delta) \hookrightarrow H^*(G_{6.i}/\Gamma, d)$.

There are the sixteen classes of groups in (3.13) and

$$\begin{aligned} i = 9, \quad b = 0; & & i = 33, \quad a = 0; & & i = 34, \quad a = 0; \\ i = 35, \quad a = -b; & & i = 89, \quad s = 0, \quad \nu_0 \neq 0; & & i = 92, \quad \nu_0 \mu_0 = 0; \\ i \in \{107, 113, \dots, 116, 125, 135\}. & & & & \end{aligned}$$

3.8.5 Decomposable solvmanifolds

The six-dimensional decomposable solvmanifolds $G/\Gamma = H_1/\Gamma_1 \times H_2/\Gamma_2$ being not a nilmanifold are contained in Table 3.5 on page 89. Using Theorem 3.2.11, one can deduce the statement about the Betti numbers. The results on the existence of a symplectic form were mostly made by Campoamor-Stursberg in [10]. He examined whether the Lie algebra admits a symplectic form. Note that in [10] the symplectic forms

$$\lambda x_{12} + \mu x_{15} + \nu x_{26} + \rho x_{34} + \sigma x_{56}, \quad \rho \neq 0, \quad \lambda\sigma \neq \mu\nu,$$

on $\mathfrak{g}_{5,14}^0 \oplus \mathfrak{g}_1$ are absent.

Since there is a monomorphism from the Lie algebra cohomology to the solvmanifold's cohomology, the existence of a symplectic form with non-exact cubic on the Lie algebra implies the existence of such an on the solvmanifold. Recall that the Lie algebra is generated by the left-invariant one-forms on the Lie group. If the Lie algebra cohomology is isomorphic to the solvmanifold's cohomology¹², one knows whether the solvmanifold is symplectic or not. Up to exact summands the symplectic forms are listed in Table 3.6 with respect to the dual of the Lie algebra's bases given in Appendix A. In the column "isom.", we mark whether there is an isomorphism of the cohomology algebras.

We do not claim that Table 3.5 contains all connected and simply-connected decomposable solvable and non-nilpotent Lie groups which admit a lattice – just those Lie groups admitting a lattice such that the corresponding solvmanifold is a product of lower-dimensional ones.

3.9 Relations with the Lefschetz property

We have seen in Chapter 2 that a compact Kähler manifold is formal, Hard Lefschetz and its odd-degree Betti numbers are even. Even if a manifold has a complex structure, these conditions are not sufficient as the following theorem which is mentioned in [41] shows. Recall, we have seen above that $G_{5,7}^{1,-1,-1}$ admits a lattice.

Theorem 3.9.1. *Let Γ be an arbitrary lattice in $G_{5,7}^{1,-1,-1}$. Then the solvmanifold $M := G_{5,7}^{1,-1,-1}/\Gamma \times S^1$ is formal, Hard Lefschetz and has even odd-degree Betti numbers. Moreover, M possesses a complex structure but it cannot be Kählerian.*

Proof. From Proposition 3.7.2.1 follows that the Lie group $G := G_{5,7}^{1,-1,-1} \times \mathbb{R}$ possesses a lattice Γ . The Chevalley-Eilenberg complex of its Lie algebra

$$\langle X_1, \dots, X_6 \mid [X_1, X_5] = X_1, [X_2, X_5] = X_2, [X_3, X_5] = -X_3, [X_4, X_5] = -X_4 \rangle$$

¹²E.g. this happens if the Lie algebra is completely solvable or if the above monomorphism must be an isomorphism by dimension reasons.

Table 3.5: Decomposable non-nil-solvmanifolds $G/\Gamma = H_1/\Gamma_1 \times H_2/\Gamma_2$

G	$b_1(G/\Gamma)$	$b_2(G/\Gamma)$	formal	sympl.	Comment
$G_{5.7}^{p,q,r} \times \mathbb{R}$	2	1	yes	no	$-1 < r < q < p < 1$, $pqr \neq 0$, $p + q + r = -1$
$G_{5.7}^{p,-p,-1} \times \mathbb{R}$	2	3	yes	yes	$p \in]0, 1[$
$G_{5.7}^{1,-1,-1} \times \mathbb{R}$	2	5	yes	yes	
$G_{5.8}^{-1} \times \mathbb{R}$	3	5	no	yes	
$G_{5.13}^{-1-2q,q,r} \times \mathbb{R}$	≥ 2	≥ 1	?	?	$q \in [-1, 0[$, $q \neq -\frac{1}{2}, r \neq 0$
$G_{5.13}^{-1,0,r} \times \mathbb{R}$	≥ 2	≥ 3	?	yes	$r \neq 0$
$G_{5.14}^0 \times \mathbb{R}$	≥ 3	≥ 5	?	yes	
$G_{5.15}^{-1} \times \mathbb{R}$	2	3	no	yes	
$G_{5.17}^{p,-p,r} \times \mathbb{R}$	≥ 2	≥ 1	?	?	$p \neq 0, r \notin \{0, \pm 1\}$,
$G_{5.17}^{p,-p,r} \times \mathbb{R}$	≥ 2	≥ 3	?	yes	$(p \neq 0, r = \pm 1)$ or $(p = 0, r \notin \{0, \pm 1\})$
$G_{5.17}^{0,0,\pm 1} \times \mathbb{R}$	≥ 2	≥ 5	?	yes	
$G_{5.18}^0 \times \mathbb{R}$	≥ 2	≥ 3	?	yes	
$G_{5.20}^{-1} \times \mathbb{R}$	3	3	yes	no	
$G_{5.26}^{0,\pm 1} \times \mathbb{R}$	≥ 3	≥ 3	?	?	
$G_{5.33}^{-1,-1} \times \mathbb{R}$	3	3	yes	no	
$G_{5.35}^{-2,0} \times \mathbb{R}$	≥ 3	≥ 3	?	?	
$G_{4.5}^{p,-p-1} \times \mathbb{R}^2$	3	3	yes	no	$p \in [-\frac{1}{2}, 0[$
$G_{4.6}^{-2p,p} \times \mathbb{R}^2$	3	3	yes	no	$p > 0$
$G_{4.8}^{-1} \times \mathbb{R}^2$	3	3	yes	no	
$G_{4.9}^0 \times \mathbb{R}^2$	3	3	yes	no	
$G_{3.4}^{-1} \times \mathbb{R}^3$	4	7	yes	yes	
$G_{3.5}^0 \times \mathbb{R}^3$	4	7	yes	yes	
$G_{3.1} \times G_{3.4}^{-1}$	3	5	no	yes	
$G_{3.1} \times G_{3.5}^0$	3	5	no	yes	
$G_{3.4}^{-1} \times G_{3.4}^{-1}$	2	3	yes	yes	
$G_{3.4}^{-1} \times G_{3.5}^0$	2	3	yes	yes	
$G_{3.5}^0 \times G_{3.5}^0$	2	3	yes	yes	

Table 3.6: Symplectic forms on $G/\Gamma = H_1/\Gamma_1 \times H_2/\Gamma_2$

\mathfrak{g}	symplectic forms	isom.
$\mathfrak{g}_{5.7}^{p,-p,-1} \oplus \mathfrak{g}_1$	$a x_{14} + b x_{23} + c x_{56}, abc \neq 0$	yes
$\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathfrak{g}_1$	$a x_{13} + b x_{14} + c x_{23} + d x_{24} + e x_{56}, e(bc - ad) \neq 0$	yes
$\mathfrak{g}_{5.8}^{-1} \oplus \mathfrak{g}_1$	$a x_{12} + b x_{15} + c x_{26} + d x_{34} + e x_{56}, d(ae - bc) \neq 0$	yes
$\mathfrak{g}_{5.13}^{-1,0,r} \oplus \mathfrak{g}_1$	$a x_{12} + b x_{34} + c x_{56}, abc \neq 0$?
$\mathfrak{g}_{5.14}^0 \oplus \mathfrak{g}_1$	$a x_{12} + b x_{15} + c x_{26} + d x_{34} + e x_{56}, d(ae - bc) \neq 0$?
$\mathfrak{g}_{5.15}^{-1} \oplus \mathfrak{g}_1$	$a(x_{14} - x_{23}) + b x_{24} + c x_{56}, abc \neq 0$	yes
$\mathfrak{g}_{5.17}^{p,-p,\pm 1} \oplus \mathfrak{g}_1$ $p \neq 0$	$a(x_{13} \pm x_{24}) + b(x_{14} \mp x_{23}) + c x_{56}, abc \neq 0$?
$\mathfrak{g}_{5.17}^{0,0,r} \oplus \mathfrak{g}_1$ $r \neq \pm 1$	$a x_{12} + b x_{34} + c x_{56}, abc \neq 0$?
$\mathfrak{g}_{5.17}^{0,0,\pm 1} \oplus \mathfrak{g}_1$	$a x_{12} + b(x_{13} \pm x_{24}) + c(x_{14} \mp x_{23}) + d x_{34} + e x_{56},$ $e(ad \mp (b^2 + c^2)) \neq 0$?
$\mathfrak{g}_{5.18}^0 \oplus \mathfrak{g}_1$	$a(x_{13} + x_{24}) + b x_{24} + c x_{56}, ac \neq 0$?
$\mathfrak{g}_{3.4}^{-1} \oplus 3\mathfrak{g}_1$	$a x_{12} + b x_{34} + c x_{35} + d x_{36} + e x_{45} + f x_{46} + g x_{56},$ $a(de - cf + bg) \neq 0$	yes
$\mathfrak{g}_{3.5}^0 \oplus 3\mathfrak{g}_1$	$a x_{12} + b x_{34} + c x_{35} + d x_{36} + e x_{45} + f x_{46} + g x_{56},$ $a(de - cf + bg) \neq 0$	yes
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.4}^{-1}$	$a x_{12} + b x_{13} + c x_{26} + d x_{36} + e x_{45}, e(ad - bc) \neq 0$	yes
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.5}^0$	$a x_{12} + b x_{13} + c x_{26} + d x_{36} + e x_{45}, e(ad - bc) \neq 0$	yes
$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_{3.4}^{-1}$	$a x_{12} + b x_{36} + c x_{45}, abc \neq 0$	yes
$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_{3.5}^0$	$a x_{12} + b x_{36} + c x_{45}, abc \neq 0$	yes
$\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_{3.5}^0$	$a x_{12} + b x_{36} + c x_{45}, abc \neq 0$	yes

is given by

$$\delta x_1 = -x_{15}, \delta x_2 = -x_{25}, \delta x_3 = x_{35}, \delta x_4 = x_{45}, \delta x_5 = \delta x_6 = 0,$$

where $\{x_1, \dots, x_6\}$ is a basis of the left-invariant one-forms on G . Since G is completely solvable, Theorem 3.2.11 (ii) enables us to compute the cohomology of M as

$$\begin{aligned} H^1(M, \mathbb{R}) &\cong \langle [x_5], [x_6] \rangle, \\ H^2(M, \mathbb{R}) &\cong \langle [x_{13}], [x_{14}], [x_{23}], [x_{24}], [x_{56}] \rangle, \\ H^3(M, \mathbb{R}) &\cong \langle [x_{135}], [x_{136}], [x_{145}], [x_{146}], [x_{235}], [x_{236}], [x_{245}], [x_{246}] \rangle, \\ H^4(M, \mathbb{R}) &\cong \langle [x_{1234}], [x_{1356}], [x_{1456}], [x_{2356}], [x_{2456}] \rangle, \\ H^5(M, \mathbb{R}) &\cong \langle [x_{12345}], [x_{12346}] \rangle. \end{aligned} \quad (3.15)$$

Let $[\omega] \in H^2(M, \mathbb{R})$ represent a symplectic form on M . A short calculation shows that there are $a, b, c, d, e \in \mathbb{R}$ with $e(bc - ad) \neq 0$ and

$$[\omega] = a[x_{13}] + b[x_{14}] + c[x_{23}] + d[x_{24}] + e[x_{56}].$$

Since $[x_5] \cup [\omega]^2 = 2(bc - de)[x_{12345}] \neq 0$ and $[x_6] \cup [\omega]^2 = 2(bc - de)[x_{12346}] \neq 0$, the homomorphism $L^2: H^1(M, \mathbb{R}) \rightarrow H^5(M, \mathbb{R})$ is an isomorphism.

In the basis (3.15), the homomorphism $L^1: H^2(M, \mathbb{R}) \rightarrow H^4(M, \mathbb{R})$ is represented by the matrix

$$\begin{pmatrix} -d & c & -b & -a & 0 \\ e & 0 & 0 & 0 & a \\ 0 & e & 0 & 0 & b \\ 0 & 0 & e & 0 & c \\ 0 & 0 & 0 & e & d \end{pmatrix}$$

which has $2e^3(ad - bc) \neq 0$ as

determinant, hence M is Hard Lefschetz.

We define an almost complex structure J on G by

$$JX_1 = X_2, \quad JX_2 = -X_1, \quad JX_3 = X_4, \quad JX_4 = -X_3, \quad JX_5 = X_6, \quad JX_6 = -X_5,$$

which induces an almost complex structure on M . It is easy to see that the Nijenhuis tensor vanishes, hence M is a complex manifold.

M is a non-toral solvmanifold which is a quotient of a completely solvable Lie group. Therefore, M cannot be Kählerian by Theorem 3.2.13. \square

The authors of [47] considered the relations between the above three properties for closed symplectic manifolds. We want to try to complete [47, Theorem 3.1 Table 1] in the case of symplectic solvmanifolds. Actually, the mentioned table deals with symplectically aspherical closed manifolds, but note that symplectic solvmanifolds are symplectically aspherical.

We start our investigations by the examination of the Lefschetz property in dimension four.

Theorem 3.9.2. *A four-dimensional symplectic solvmanifold is not (Hard) Lefschetz if and only if it is a non-toral nilmanifold. Especially, the (Hard) Lefschetz property is independent of the choice of the symplectic form.*

Proof. By Theorem 3.6.2, there are five classes of four-dimensional symplectic solvmanifolds. Three of them are nilmanifolds and satisfy the claim by Corollary 3.1.10.

There remain two non-nilmanifolds to consider. We start with a quotient M of the Lie group which has $\mathfrak{g}_{3,1}^{-1} \oplus \mathfrak{g}_1$ as Lie algebra, see Table A.1. The Lie group is completely solvable, hence the Lie algebra cohomology is isomorphic to the solvmanifold's cohomology. If x_1, \dots, x_4 denote the left-invariant one-forms which are dual to the basis given in Table A.1, one computes

$$\begin{aligned} H^1(M, \mathbb{R}) &\cong \langle [x_3], [x_4] \rangle, \\ H^2(M, \mathbb{R}) &\cong \langle [x_{12}], [x_{34}] \rangle, \\ H^3(M, \mathbb{R}) &\cong \langle [x_{123}], [x_{124}] \rangle. \end{aligned} \tag{3.16}$$

The class representing a symplectic form must be of the form $[a x_{12} + b x_{34}]$ with $a, b \neq 0$ and obviously, the Lefschetz map with respect to this class is an isomorphism.

Now, consider a solvmanifold G/Γ such that the Lie algebra of G is $\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1$ and $b_1(G/\Gamma) = 2$. A short computation yields that the Lie algebra cohomology of $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$ is the same as in (3.16). Since G/Γ is compact and parallelisable, we see further $b_i(G/\Gamma) = 2$ for $i \in \{1, 2, 3\}$, and Theorem 3.2.11 (i) implies that (3.16) also gives the cohomology of G/Γ . We have yet seen that a symplectic four-manifold with this cohomology is Hard Lefschetz. \square

Denote KT “the” four-dimensional symplectic nilmanifold with $b_1(KT) = 3$. We have seen that KT is not formal and not Lefschetz. Its square has the following properties:

Theorem 3.9.3 ([47]). *$KT \times KT$ is not formal, not Lefschetz and has even odd-degree Betti numbers.* \square

Next, we are looking for an example of a formal manifold that is not Lefschetz and has even odd degree Betti numbers resp. an odd odd degree Betti number.

Theorem 3.9.4. *The Lie group $G_{6,78}$ admits a lattice Γ , see above. $M := G_{6,78}/\Gamma$ is a formal solvmanifold with $b_1(M) = 1$ that admits a symplectic form ω such that (M, ω) is not Hard Lefschetz. Moreover, $(M \times M, \omega \times \omega)$ is a formal symplectic manifold with even odd degree Betti numbers that is not Hard Lefschetz.*

Proof. By Theorem 3.8.3.4, M is a formal symplectic manifold with Betti numbers $b_1(M) = b_2(M) = 1$. Note that this implies that $M \times M$ is symplectic and formal (the latter property by Proposition 1.1.7).

Corollary 2.1.4 forces M to be not Lefschetz and since [27, Proposition 4.2] says that a product is Lefschetz if and only if both factors are Lefschetz, $M \times M$ is not Lefschetz.

M is a six-dimensional solvmanifold and so it is parallelisable. Hence the fact $b_0(M) = b_1(M) = b_2(M) = 1$ implies $b_3(M) = 2$. This and Poincaré Duality imply $b_1(M \times M) = b_{11}(M \times M) = 2$, $b_3(M \times M) = b_9(M \times M) = 6$ and $b_5(M \times M) = b_7(M \times M) = 4$. \square

In 1990, Benson and Gordon [4, Example 3] constructed an eight-dimensional non-exact symplectic and completely solvable Lie algebra that does not satisfy the Hard Lefschetz property, but they did not know whether the corresponding connected and simply-connected Lie group G^{BG} admits a lattice.

Fernández, de León and Saralegui computed in [26, Proposition 3.2] the minimal model of the complex of the left-invariant differential forms on G^{BG} . It is formal and its cohomology of odd degree is even-dimensional. If G^{BG} admits a lattice, by complete solvability, the corresponding solvmanifold would be a symplectic and formal manifold with even odd degree Betti numbers that violates the Hard Lefschetz property.

In 2000, Tralle [74] claimed that a lattice does not exist but Sawai and Yamada noted 2005 Tralle’s proof would contain calculatory errors and constructed a lattice [69, Theorem 1]. This proves the next theorem.

Theorem 3.9.5. *There exists an eight-dimensional symplectic and formal solvmanifold M^{BG} with even odd degree Betti numbers that is not Hard Lefschetz.* \square

We sum up the above results in Table 3.7. It is an enlargement of [47, Theorem 3.1 Table 1].

Table 3.7: Relations of the Kähler properties

Formality	Hard Lefschetz	$b_{2i+1} \equiv 0(2)$	Example
yes	yes	yes	Kähler
yes	yes	no	impossible
yes	no	yes	$M^{BG}, G_{6.78}/\Gamma \times G_{6.78}/\Gamma$
yes	no	no	$G_{6.78}/\Gamma$
no	yes	yes	?
no	yes	no	impossible
no	no	yes	$KT \times KT$
no	no	no	KT

Unfortunately, the missing example does not arise among the six-dimensional solvmanifolds that possess the same cohomology as the corresponding Lie algebra. In order to see this, one has to examine which of them satisfy the (Hard) Lefschetz property. We briefly mention the results.

By Corollary 2.1.4, a manifold with odd first Betti number cannot be Lefschetz. We now examine such indecomposable solvmanifolds whose first Betti number is even; in the completely solvable case, these are quotients of $G_{6,3}^{-1}$, $G_{6,21}^0$ and $G_{6,54}^{-1}$.

The proof of the next two propositions is done analogous as that of Theorem 3.9.1. By complete solvability, we know the solvmanifolds' cohomology and all possible symplectic forms were determined in the proof of Proposition 3.8.4.1. Therefore, one can compute the image of the Lefschetz maps.

Proposition 3.9.6. *Let a lattice in $G_{6,3}^{-1}$ or $G_{6,21}^0$ be given. (We have seen above that such exists.) Then the corresponding (non-formal) symplectic solvmanifold (with $b_1 = 2$, $b_2 = 3$) is not Lefschetz, independent of the choice of the symplectic form. \square*

Proposition 3.9.7. *Let a lattice in $G_{6,54}^{-1}$ be given. (Such exists by Proposition 3.8.4.6.) The corresponding (non-formal) symplectic solvmanifold (with $b_1 = 2$, $b_2 = 5$) is Lefschetz but not Hard Lefschetz, independent of the choice of the symplectic form. \square*

Remark. The existence of a lattice in $G_{6,54}^{-1}$ was proven by Fernández, de León and Saralegui in [26]. They also computed the Betti numbers of the corresponding solvmanifold, showed that it is not formal and does not satisfy the Hard Lefschetz property with respect to a certain symplectic form. Moreover, Fernández and Muñoz proved in [27, Example 3] that the manifold is Lefschetz. (Analogous calculations work for other symplectic forms.)

In the non-completely solvable case, the situation becomes a little more complicated. If we are willing to make a statement about the Lefschetz property, we have to know the cohomology and need therefore assumptions on the Betti numbers.

Proposition 3.9.8. *If there is a lattice in one of the non-completely solvable groups $G_{6,i}^{0,0}$, $i \in \{10, 36, 70\}$ resp. $G_{6,118}^{0,\pm 1,-1}$ such that the cohomology of the corresponding solvmanifold M_i is isomorphic to the Lie algebra cohomology of $\mathfrak{g}_{6,i}$ (i.e. the cohomology is as small as possible), then one computes that the following hold, independent of the choice of the symplectic forms provided by Proposition 3.8.4.1:*

- M_{10} and M_{36} are not formal and not Lefschetz.
- M_{70} is formal and Lefschetz but not Hard Lefschetz.
- M_{118} is formal and Hard Lefschetz.

(The statements on formality follow from the propositions at the end of Section 3.8.4.) \square

Finally, we consider the decomposable symplectic solvmanifolds listed in Table 3.5.

Proposition 3.9.9. *Let $G/\Gamma = H_1/\Gamma_1 \times H_2/\Gamma_2$ be one of the symplectic solvmanifolds listed in Table 3.5 such that in the corresponding row of the table arises no \geq -sign.*

Then G/Γ is formal if and only if it is Hard Lefschetz (independent of the special choice of the symplectic form).

Sketch of the proof. One has an isomorphism from the Lie algebra cohomology to the solvmanifold's cohomology for each manifold as in the theorem. Then an explicit calculation as in the proof of Theorem 3.9.1 yields that the Hard Lefschetz manifolds among the considered are exactly the formal ones.

Note, if b_1 is not even, the claim follows directly from Theorem 2.1.3. \square

Remark. $G_{5,15}^{-1}/\Gamma_1 \times S^1$ is Lefschetz. The other manifolds in the last proposition are even not Lefschetz if they are not Hard Lefschetz.

$G_{5,15}^{-1}/\Gamma_1 \times S^1$ is a Lefschetz manifold that is not formal and has even odd degree Betti numbers.

A similar result as the last proposition holds for the manifolds in Table 3.5 such that in the corresponding row of the table arises a \geq -sign. But we again must make an assumption that enables us to compute the whole cohomology.

Proposition 3.9.10.

(i) *Let $M = G_{5,i}/\Gamma \times \mathbb{R}/\mathbb{Z}$ be a symplectic manifold such that one of the following conditions holds:*

- a) $i = 13$ with $q = 0$ and $b_1(M) = 2$ as well as $b_2(M) = 3$,
- b) $i = 17$ with $p \neq 0$, $r = \pm 1$ and $b_1(M) = 2$ as well as $b_2(M) = 3$,
- c) $i = 17$ with $p = 0$, $r \notin \{0, \pm 1\}$ and $b_1(M) = 2$ as well as $b_2(M) = 3$,
- d) $i = 17$ with $p = 0$, $r = \pm 1$ and $b_1(M) = 2$ as well as $b_2(M) = 5$.

Then M is formal and Hard Lefschetz (independent of the special choice of the symplectic form).

(ii) *Let Γ be a lattice in $G_{5,14}^0$ such that $M = G_{5,14}^0/\Gamma \times \mathbb{R}/\mathbb{Z}$ satisfies $b_1(M) = 3$ and $b_2(M) = 5$.*

Then M is not formal and not Lefschetz (independent of the special choice of the symplectic form).

(iii) *Let Γ be a lattice in $G_{5,18}^0$ such that $M = G_{5,18}^0/\Gamma \times \mathbb{R}/\mathbb{Z}$ satisfies $b_1(M) = 2$ and $b_2(M) = 3$.*

Then M is not formal and Lefschetz but not Hard Lefschetz (independent of the special choice of the symplectic form). \square

Chapter 4

Rational Homotopy and Minimal Models

In general, it is very difficult to calculate the homotopy groups $\pi_k(X)$ of a given topological space X . However, if one is willing to forget the torsion, with certain assumptions on X , the rational homotopy groups $\pi_k(X) \otimes \mathbb{Q}$ can be determined by the theory of minimal models.

4.1 PL forms

In order to relate minimal models to rational homotopy theory, we need a differential graded algebra over \mathbb{Q} to replace the de Rham algebra.

Let Δ^n be a standard simplex in \mathbb{R}^{n+1} and $(\Omega_{PL}(\Delta^n), d)$ the restriction to Δ^n of all differential forms in \mathbb{R}^{n+1} which can be written as $\sum P_{i_1 \dots i_k} dx_{i_1} \dots dx_{i_k}$, where $P_{i_1 \dots i_k} \in \mathbb{Q}[x_1, \dots, x_{n+1}]$ together with multiplication and differential induced by \mathbb{R}^{n+1} .

Let $X = \{(\sigma_i)_{i \in I}\}$ be a path-connected simplicial complex. Set for $k \in \mathbb{Z}$

$$\Omega_{PL}^k(X) := \{(\alpha_i)_{i \in I} \mid \alpha_i \in \Omega_{PL}^k(\sigma_i) \wedge (\sigma_i \subset \partial\sigma_j \Rightarrow \alpha_j|_{\sigma_i} = \alpha_i)\},$$

and $\Omega_{PL}(X) := \bigoplus_{k \in \mathbb{Z}} \Omega_{PL}^k(X)$. It can be verified that the set $\Omega_{PL}(X)$ of so-called *PL forms* is a differential graded algebra over \mathbb{Q} if we use the multiplication and the differential on forms componentwise.

Analogous to the usual result for the de Rham complex, we have:

Theorem 4.1.1 ([63, Theorem 1.1.4]). *If X is a path-connected simplicial complex, then there is an isomorphism $H^*(\Omega_{PL}(X), d) \cong H^*(X, \mathbb{Q})$.* \square

For such a simplicial complex X , we define the (\mathbb{Q}) -minimal model $\mathcal{M}_{X, \mathbb{Q}}$ of X to be the minimal model of $(\Omega_{PL}(X), d)$. Its relation to the minimal model of a smooth manifold (see Chapter 1) is given by the following theorem.

Theorem 4.1.2 ([63, Theorem 1.3.9]). *Let M be a connected smooth manifold. Then there is an isomorphism $\mathcal{M}_{M, \mathbb{Q}} \otimes \mathbb{R} \cong \mathcal{M}_M$.* \square

4.2 Nilpotent spaces

Already in his paper [72], Sullivan shows that for nilpotent spaces, there is a correspondence between the minimal model and the rational homotopy. To state this result, we need the notion of a nilpotent space resp. nilpotent module.

Let G be a group, H be a G -module, $\Gamma_G^0 H := H$ and

$$\Gamma_G^{i+1} H := \langle g.h - h \mid g \in G \wedge h \in \Gamma_G^i H \rangle \subset \Gamma_G^i H$$

for $i \in \mathbb{N}$.

Then, H is called a *nilpotent module* if there is $n_0 \in \mathbb{N}$ such that $\Gamma_G^{n_0} H = \{1\}$.

We recall the natural π_1 -module structure of the higher homotopy groups π_n of a topological space. For instance, let (X, x_0) be a pointed space with universal cover (\tilde{X}, \tilde{x}_0) . It is well known that $\pi_1(X, x_0) \cong D(\tilde{X})$, the group of deck transformations of the universal covering. Now, because \tilde{X} is simply-connected, every free homotopy class of self-maps of \tilde{X} determines uniquely a class of basepoint preserving self-maps of \tilde{X} (see e.g. [43, Proposition 4.A.2]). This means that to every homotopy class of deck transformations corresponds a homotopy class of basepoint preserving self-maps (which are, in fact, homotopy equivalences) $(\tilde{X}, \tilde{x}_0) \rightarrow (\tilde{X}, \tilde{x}_0)$. These maps provide induced automorphisms of homotopy groups $\pi_n(\tilde{X}, \tilde{x}_0) \cong \pi_n(X, x_0)$ ($n > 1$) and this whole process then provides an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Definition 4.2.1. A path-connected topological space X whose universal covering exists is called *nilpotent* if for $x_0 \in X$ the fundamental group $\pi_1(X, x_0)$ is a nilpotent group and the higher homotopy groups $\pi_n(X, x_0)$ are nilpotent $\pi_1(X, x_0)$ -modules for all $n \in \mathbb{N}$, $n \geq 2$. Note, the definition is independent of the choice of the base point.

Example.

- (i) Simply-connected spaces are nilpotent.
- (ii) S^1 is nilpotent.
- (iii) The cartesian product of two nilpotent spaces is nilpotent. Therefore, all tori are nilpotent.
- (iv) The Klein bottle is not nilpotent.
- (v) $P^n(\mathbb{R})$ is nilpotent if and only if $n \equiv 1(2)$.

Proof. (i) - (iv) are obvious and (v) can be found in Hilton's book [46] on page 165. \square

The main theorem on the rational homotopy of nilpotent spaces is the following.

Theorem 4.2.2. *Let X be a path-connected nilpotent CW-complex with finitely generated homotopy groups. If $\mathcal{M}_{X,\mathbb{Q}} = \bigwedge V$ denotes the minimal model, then for all $k \in \mathbb{N}$ with $k \geq 2$ holds:*

$$\mathrm{Hom}_{\mathbb{Z}}(\pi_k(X), \mathbb{Q}) \cong V^k$$

Using another approach to minimal models (via localisation of spaces and Postnikov towers), this theorem is proved for example in [50]. The proof that we shall give here is new to the author's knowledge. We will show the following more general result mentioned (but not proved) by Halperin in [37].

Theorem 4.2.3. *Let X be a path-connected triangulable topological space whose universal covering exists. Denote by $\mathcal{M}_{X,\mathbb{Q}} = \bigwedge V$ the minimal model and assume that*

- (i) *each $\pi_k(X)$ is a finitely generated nilpotent $\pi_1(X)$ -module for $k \geq 2$ and*
- (ii) *the \mathbb{Q} -minimal model for $K(\pi_1(X), 1)$ has no generators in degrees greater than one.*

Then for each $k \geq 2$ there is an isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\pi_k(X), \mathbb{Q}) \cong V^k$.

Remark. The homotopy groups of a compact nilpotent smooth manifold are finitely generated:

By [46, Satz 7.22], a nilpotent space has finitely generated homotopy if and only if it has finitely generated homology with \mathbb{Z} -coefficients. The latter is satisfied for compact spaces. \square

The main tool for the proof of the above theorems is a consequence of the fundamental theorem of Halperin [37]. In the next section, we quote it and use it to prove Theorems 4.2.2 and 4.2.3.

4.3 The Halperin-Grivel-Thomas theorem

To state the theorem, let us recall a basic construction for fibrations.

Let $\pi: E \rightarrow B$ be a fibration with path-connected basis B . Therefore, all fibers $F_b = \pi^{-1}(\{b\})$ are homotopy equivalent to a fixed fiber F since each path γ in B lifts to a homotopy equivalence $L_\gamma: F_{\gamma(0)} \rightarrow F_{\gamma(1)}$ between the fibers over the endpoints of γ . In particular, restricting the paths to loops at a basepoint of B we obtain homotopy equivalences $L_\gamma: F \rightarrow F$ for F the fibre over the basepoint b_0 . One can show that this induces a natural $\pi_1(B, b_0)$ -module structure on $H^*(F, \mathbb{Q})$.

Theorem 4.3.1 ([63, Theorem 1.4.4]). *Let F, E, B be path-connected triangulable topological spaces and $F \rightarrow E \rightarrow B$ a fibration such that $H^n(F, \mathbb{Q})$ is a nilpotent $\pi_1(B, b_0)$ -module for $n \in \mathbb{N}_+$. The fibration induces a sequence*

$$(\Omega_{PL}(B), d_B) \longrightarrow (\Omega_{PL}(E), d_E) \longrightarrow (\Omega_{PL}(F), d_F)$$

of differential graded algebras. Suppose that $H^*(F, \mathbb{Q})$ or $H^*(B, \mathbb{Q})$ is of finite type.

Then there is a quasi-isomorphism $\Psi: (\mathcal{M}_{B, \mathbb{Q}} \otimes \mathcal{M}_{F, \mathbb{Q}}, D) \rightarrow (\Omega_{PL}(E), d_E)$ making the following diagram commutative:

$$\begin{array}{ccccc}
 (\Omega_{PL}(B), d_B) & \longrightarrow & (\Omega_{PL}(E), d_E) & \longrightarrow & (\Omega_{PL}(F), d_F) \\
 \uparrow \rho_B & & \uparrow \Psi & & \uparrow \rho_F \\
 (\mathcal{M}_{B, \mathbb{Q}}, D_B) & \hookrightarrow & (\mathcal{M}_{B, \mathbb{Q}} \otimes \mathcal{M}_{F, \mathbb{Q}}, D) & \longrightarrow & (\mathcal{M}_{F, \mathbb{Q}}, D_F)
 \end{array}$$

Furthermore, the left and the right vertical arrows are the minimal models. Moreover, if $\mathcal{M}_{F, \mathbb{Q}} = \bigwedge V_F$, there is an ordered basis $\{v_i^F \mid i \in I\}$ of V_F such that for all $i, j \in I$ holds $D(v_i^F) \in \mathcal{M}_{B, \mathbb{Q}} \otimes (\mathcal{M}_{F, \mathbb{Q}})_{<v_i^F}$ and $(v_i^F < v_j^F \Rightarrow |v_i^F| \leq |v_j^F|)$. \square

Remark. In general, $(\mathcal{M}_{B, \mathbb{Q}} \otimes \mathcal{M}_{F, \mathbb{Q}}, D)$ is not a minimal differential graded algebra and $D|_{\mathcal{M}_{F, \mathbb{Q}}} \neq D_F$ is possible.

We need some further preparations for the proofs of the above theorems. The first is a reformulation of the results 3.8 – 3.10 in [46]. It justifies the statement of the next theorem.

Proposition 4.3.2. *Let G be a finitely generated nilpotent group. Then the set $T(G)$ of torsion elements of G is a finite normal subgroup of G and $G/T(G)$ is finitely generated.* \square

Theorem 4.3.3. *Let G be a finite generated nilpotent group and denote by $T(G)$ its finite normal torsion group.*

Then $K(G, 1)$ and $K(G/T(G), 1)$ share their minimal model.

Proof. Since $T(G)$ is finite and \mathbb{Q} is a field, we get from [22, Section 4.2] $H^n(K(T(G), 1), \mathbb{Q}) = \{0\}$ for $n \in \mathbb{N}_+$. The construction of the minimal model in the proof of Theorem 1.1.2 implies that $\mathcal{M}_{K(T(G), 1), \mathbb{Q}}$ has no generators of degree greater than zero. Now, the theorem follows from the preceding one, applied to the fibration $K(T(G), 1) \rightarrow K(G, 1) \rightarrow K(G/T(G), 1)$. \square

Lemma 4.3.4. *Let X be topological space with universal covering $\mathfrak{p}: \tilde{X} \rightarrow X$.*

Then, up to weak homotopy equivalence of the total space, there is a fibration $\tilde{X} \rightarrow X \rightarrow K := K(\pi_1(X), 1)$. Moreover, for a class $[\gamma] \in \pi_1(K) \cong \pi_1(X)$ the homotopy equivalences $L_{[\gamma]}: \tilde{X} \rightarrow \tilde{X}$ described at the beginning of this section are given by the corresponding deck transformations of \mathfrak{p} .

Proof. Denote by $\pi: E \rightarrow K(\pi_1(X), 1)$ the universal principal $\pi_1(X)$ -bundle. Regard on $E \times \tilde{X}$ the diagonal $\pi_1(X)$ -action. Then, the fibre bundle

$$\tilde{X} \longrightarrow ((E \times \tilde{X})/\pi_1(X)) \longrightarrow K$$

has the desired properties. \square

Proof of Theorem 4.2.3:

Let X be as in the statement of the theorem. For simply-connected spaces, the theorem was proven in [23, Theorem 15.11]. Now, the idea is to use this result and to consider the universal cover $\mathbf{p}: \tilde{X} \rightarrow X$. Denote by $\mathcal{M}_{\tilde{X}, \mathbb{Q}} = \bigwedge \tilde{V}$ and $\mathcal{M}_{X, \mathbb{Q}} = \bigwedge V$ the minimal models. We shall show

$$\forall_{k \geq 2} V^k \cong \tilde{V}^k. \quad (4.1)$$

This and the truth of the theorem for simply-connected spaces implies then the general case

$$\forall_{k \geq 2} V^k \cong \tilde{V}^k \cong \text{Hom}_{\mathbb{Z}}(\pi_k(\tilde{X}), \mathbb{Q}) = \text{Hom}_{\mathbb{Z}}(\pi_k(X), \mathbb{Q}).$$

It remains to show (4.1): Since X is triangulable, X and \tilde{X} can be seen as CW-complexes. Therefore, up to weak homotopy, there is the following fibration of CW-complexes

$$\tilde{X} \longrightarrow X \xrightarrow{\pi} K(\pi_1(X), 1) =: K.$$

We prove below:

$$H^*(\tilde{X}, \mathbb{Q}) \text{ is of finite type.} \quad (4.2)$$

$$H^*(\tilde{X}, \mathbb{Q}) \text{ is a nilpotent } \pi_1(X)\text{-module.} \quad (4.3)$$

Then Theorem 4.3.1 implies the existence of a quasi-isomorphism ρ such that the following diagram commutes:

$$\begin{array}{ccccc} (\Omega_{PL}(K), d_K) & \longrightarrow & (\Omega_{PL}(X), d_X) & \longrightarrow & (\Omega_{PL}(\tilde{X}), d_{\tilde{X}}) \\ \uparrow \rho_K & & \uparrow \rho & & \uparrow \rho_{\tilde{X}} \\ (\mathcal{M}_{K, \mathbb{Q}}, D_K) & \hookrightarrow & (\mathcal{M}_{K, \mathbb{Q}} \otimes \mathcal{M}_{\tilde{X}, \mathbb{Q}}, D) & \longrightarrow & (\mathcal{M}_{\tilde{X}, \mathbb{Q}}, D_{\tilde{X}}) \end{array}$$

Finally, we shall see

$$(\mathcal{M}_{K, \mathbb{Q}} \otimes \mathcal{M}_{\tilde{X}, \mathbb{Q}}, D) \text{ is a minimal differential graded algebra} \quad (4.4)$$

and this implies (4.1) since \mathcal{M}_K has no generators of degree greater than one by assumption (ii).

We still have to prove (4.2) - (4.4):

By assumption (i), $\pi_k(X) = \pi_k(\tilde{X})$ is finitely generated for $k \geq 2$. Since simply-connected spaces are nilpotent, [46, Satz 7.22] implies the finite generation of $H_*(\tilde{X}, \mathbb{Z})$ and (4.2) follows.

(4.3) is the statement of Theorem 2.1 (i) \Rightarrow (ii) in [45] – applied to the action of $\pi_1(X)$ on $\pi_i(\tilde{X})$.

ad (4.4): By assumption (ii), \mathcal{M}_K has no generators in degrees greater than one, i.e. $\mathcal{M}_{K,\mathbb{Q}} = \bigwedge\{v_i \mid i \in I\}$ with $|v_i| = 1$. The construction of the minimal model in the proof of Theorem 1.1.2 implies that the minimal model of a simply-connected space has no generators in degree one, i.e. $\mathcal{M}_{\tilde{X},\mathbb{Q}} = \bigwedge\{w_j \mid j \in J\}$ with $|w_j| > 1$. We expand the well orderings of I and J to a well ordering of their union by $\forall i \in I \forall j \in J \ i < j$. Theorem 4.3.1 implies that $D(w_j)$ contains only generators which are ordered before w_j . Trivially, $D(v_i)$ also has this property, so we have shown (4.4) and the theorem is proved. \square

Proof of Theorem 4.2.2:

Let X be a path-connected nilpotent CW-complex with finitely generated fundamental group and finitely generated homotopy. By Theorem 4.2.3, we have to show that the minimal model of $K(\pi_1(X), 1)$ has no generators in degrees greater than one. Theorem 4.3.3 implies that it suffices to show that $K(\pi_1(X)/T, 1)$ has this property, where T denotes the torsion group of $\pi_1(X)$. $\Gamma := \pi_1(X)/T$ is a finitely generated nilpotent group without torsion. By [66, Theorem 2.18], Γ can be embedded as a lattice in a connected and simply-connected nilpotent Lie group G . Therefore, the nilmanifold G/Γ is a $K(\Gamma, 1)$ and from Theorem 3.2.11 follows that its minimal model has no generators in degrees greater than one. \square

Appendix A

Lists of Lie Algebras

In Table A.1, we give the isomorphism classes of Lie algebras of the simply-connected solvable Lie groups up to dimension four that possesses lattices. The designation $\mathfrak{g}_{i,j}$ means the j -th indecomposable solvable Lie algebra of dimension i . The choice of the integer j bases on the notation of [56]. The superscripts, if any, give the values of the continuous parameters on which the algebra depends. (We do not claim that the corresponding Lie groups admit a lattice for all parameters. We just know that there exist such for certain parameters!)

Table A.1: Solvmanifolds up to dimension four

	$[X_i, X_j]$	cpl. solv.
\mathfrak{g}_1		abelian
$2\mathfrak{g}_1$		abelian
$3\mathfrak{g}_1$		abelian
$\mathfrak{g}_{3.1}$	$[X_2, X_3] = X_1$	nilpotent
$\mathfrak{g}_{3.4}^{-1}$	$[X_1, X_3] = X_1, [X_2, X_3] = -X_2$	yes
$\mathfrak{g}_{3.5}^0$	$[X_1, X_3] = -X_2, [X_2, X_3] = X_1$	no
$4\mathfrak{g}_1$		abelian
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$	$[X_2, X_3] = X_1$	nilpotent
$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_1$	$[X_1, X_3] = X_1, [X_2, X_3] = -X_2$	yes
$\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$	$[X_1, X_3] = -X_2, [X_2, X_3] = X_1$	no
$\mathfrak{g}_{4.1}$	$[X_2, X_4] = X_1, [X_3, X_4] = X_2$	nilpotent
$\mathfrak{g}_{4.5}^{p, -p-1}$	$[X_1, X_4] = X_1, [X_2, X_4] = pX_2,$ $[X_3, X_4] = (-p-1)X_3, -\frac{1}{2} \leq p < 0$	yes
$\mathfrak{g}_{4.6}^{-2p, p}$	$[X_1, X_4] = -2pX_1, [X_2, X_4] = pX_2 - X_3,$ $[X_3, X_4] = X_2 + pX_3, p > 0$	no
$\mathfrak{g}_{4.8}^{-1}$	$[X_2, X_3] = X_1, [X_2, X_4] = X_2, [X_3, X_4] = -X_3$	yes
$\mathfrak{g}_{4.9}^0$	$[X_2, X_3] = X_1, [X_2, X_4] = -X_3, [X_3, X_4] = X_2$	no

The set of non-isomorphic five dimensional nilpotent Lie algebras is exhausted by three types of decomposable algebras and six indecomposables which are listed in Table A.2. The designation is taken from [57].

Table A.2: 5-dimensional nilpotent algebras

	$[X_i, X_j]$
$5\mathfrak{g}_1$	abelian
$\mathfrak{g}_{3.1} \oplus 2\mathfrak{g}_1$	$[X_2, X_3] = X_1$
$\mathfrak{g}_{4.1} \oplus \mathfrak{g}_1$	$[X_2, X_4] = X_1, [X_3, X_4] = X_2$
$\mathfrak{g}_{5.1}$	$[X_3, X_5] = X_1, [X_4, X_5] = X_2$
$\mathfrak{g}_{5.2}$	$[X_2, X_5] = X_1, [X_3, X_5] = X_2, [X_4, X_5] = X_3$
$\mathfrak{g}_{5.3}$	$[X_2, X_4] = X_3, [X_2, X_5] = X_1, [X_4, X_5] = X_2$
$\mathfrak{g}_{5.4}$	$[X_2, X_4] = X_1, [X_3, X_5] = X_1$
$\mathfrak{g}_{5.5}$	$[X_3, X_4] = X_1, [X_2, X_5] = X_1, [X_3, X_5] = X_2$
$\mathfrak{g}_{5.6}$	$[X_3, X_4] = X_1, [X_2, X_5] = X_1, [X_3, X_5] = X_2, [X_4, X_5] = X_3$

There are 24 classes of solvable and non-nilpotent decomposable Lie algebras in dimension five. The unimodular among them are the ones in Table A.3.

Table A.3: 5-dimensional decomposable unimodular non-nilpotent algebras

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{3.4}^{-1} \oplus 2\mathfrak{g}_1$	$[X_1, X_3] = X_1, [X_2, X_3] = -X_2$	yes
$\mathfrak{g}_{3.5}^0 \oplus 2\mathfrak{g}_1$	$[X_1, X_3] = -X_2, [X_2, X_3] = X_1$	no
$\mathfrak{g}_{4.2}^{-2} \oplus \mathfrak{g}_1$	$[X_1, X_4] = -2X_1, [X_2, X_4] = X_2,$ $[X_3, X_4] = X_2 + X_3$	yes
$\mathfrak{g}_{4.5}^{p, -p-1} \oplus \mathfrak{g}_1$	$[X_1, X_4] = X_1, [X_2, X_4] = pX_2,$ $[X_3, X_4] = (-p-1)X_3, -\frac{1}{2} \leq p < 0$	yes
$\mathfrak{g}_{4.6}^{-2p, p} \oplus \mathfrak{g}_1$	$[X_1, X_4] = -2pX_1, [X_2, X_4] = pX_2 - X_3,$ $[X_3, X_4] = X_2 + pX_3, p > 0$	no
$\mathfrak{g}_{4.8}^{-1} \oplus \mathfrak{g}_1$	$[X_2, X_3] = X_1, [X_2, X_4] = X_2, [X_3, X_4] = -X_3$	yes
$\mathfrak{g}_{4.9}^0 \oplus \mathfrak{g}_1$	$[X_2, X_3] = X_1, [X_2, X_4] = -X_3, [X_3, X_4] = X_2$	no

Except for $\mathfrak{g}_{4.2} \oplus \mathfrak{g}_1$, to each class of algebras there is a connected and simply-connected solvable Lie group admitting a lattice and has a Lie algebra belonging to the class.

Mubarakzjanov's list in [57] contains 33 classes of five-dimensional indecomposable non-nilpotent solvable Lie algebras, namely $\mathfrak{g}_{5.7}, \dots, \mathfrak{g}_{5.39}$. We list the unimodular among them in Tables A.4 to A.7.

Note that there is a minor misprint in [57] which has been corrected in the list below.

Table A.4: 5-dimensional indecomposable unimodular almost abelian algebras

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{5.7}^{p,q,r}$	$[X_1, X_5] = X_1, [X_2, X_5] = pX_2,$ $[X_3, X_5] = qX_3, [X_4, X_5] = rX_4,$ $-1 \leq r \leq q \leq p \leq 1, pqr \neq 0, p + q + r = -1$	yes
$\mathfrak{g}_{5.8}^{-1}$	$[X_2, X_5] = X_1, [X_3, X_5] = X_3, [X_4, X_5] = -X_4,$	yes
$\mathfrak{g}_{5.9}^{p,-2-p}$	$[X_1, X_5] = X_1, [X_2, X_5] = X_1 + X_2, [X_3, X_5] = pX_3,$ $[X_4, X_5] = (-2 - p)X_4, p \geq -1$	yes
$\mathfrak{g}_{5.11}^{-3}$	$[X_1, X_5] = X_1, [X_2, X_5] = X_1 + X_2,$ $[X_3, X_5] = X_2 + X_3, [X_4, X_5] = -3X_4,$	yes
$\mathfrak{g}_{5.13}^{-1-2q,q,r}$	$[X_1, X_5] = X_1, [X_2, X_5] = (-1 - 2q)X_2,$ $[X_3, X_5] = qX_3 - rX_4, [X_4, X_5] = rX_3 + qX_4,$ $-1 \leq q \leq 0, q \neq -\frac{1}{2}, r \neq 0$	no
$\mathfrak{g}_{5.14}^0$	$[X_2, X_5] = X_1, [X_3, X_5] = -X_4, [X_4, X_5] = X_3$	no
$\mathfrak{g}_{5.15}^{-1}$	$[X_1, X_5] = X_1, [X_2, X_5] = X_1 + X_2,$ $[X_3, X_5] = -X_3, [X_4, X_5] = X_3 - X_4$	yes
$\mathfrak{g}_{5.16}^{-1,q}$	$[X_1, X_5] = X_1, [X_2, X_5] = X_1 + X_2,$ $[X_3, X_5] = -X_3 - qX_4, [X_4, X_5] = qX_3 - X_4,$ $q \neq 0$	no
$\mathfrak{g}_{5.17}^{p,-p,r}$	$[X_1, X_5] = pX_1 - X_2, [X_2, X_5] = X_1 + pX_2,$ $[X_3, X_5] = -pX_3 - rX_4, [X_4, X_5] = rX_3 - pX_4,$ $r \neq 0$	no
$\mathfrak{g}_{5.18}^0$	$[X_1, X_5] = -X_2, [X_2, X_5] = X_1,$ $[X_3, X_5] = X_1 - X_4, [X_4, X_5] = X_2 + X_3$	no

Table A.5: 5-dimensional indecomposable unimodular algebras with nilradical

 $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{5.19}^{p,-2p-2}$	$[X_2, X_3] = X_1, [X_1, X_5] = (1 + p)X_1, [X_2, X_5] = X_2,$ $[X_3, X_5] = pX_3, [X_4, X_5] = (-2p - 2)X_4, p \neq -1$	yes
$\mathfrak{g}_{5.20}^{-1}$	$[X_2, X_3] = X_1, [X_2, X_5] = X_2, [X_3, X_5] = -X_3,$ $[X_4, X_5] = X_1$	yes
$\mathfrak{g}_{5.23}^{-4}$	$[X_2, X_3] = X_1, [X_1, X_5] = 2X_1, [X_2, X_5] = X_2 + X_3,$ $[X_3, X_5] = X_3, [X_4, X_5] = -4X_4$	yes
$\mathfrak{g}_{5.25}^{p,4p}$	$[X_2, X_3] = X_1, [X_1, X_5] = 2pX_1, [X_2, X_5] = pX_2 + X_3,$ $[X_3, X_5] = -X_2 + pX_3, [X_4, X_5] = -4pX_4, p \neq 0$	no
$\mathfrak{g}_{5.26}^{0,\varepsilon}$	$[X_2, X_3] = X_1, [X_2, X_5] = X_3, [X_3, X_5] = -X_2,$ $[X_4, X_5] = \varepsilon X_1, \varepsilon = \pm 1$	no
$\mathfrak{g}_{5.28}^{-\frac{3}{2}}$	$[X_2, X_3] = X_1, [X_1, X_5] = -\frac{1}{2}X_1, [X_2, X_5] = -\frac{3}{2}X_2,$ $[X_3, X_5] = X_3 + X_4, [X_4, X_5] = X_4$	yes

Table A.6: 5-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{4.1}$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{5.30}^{-\frac{4}{3}}$	$[X_2, X_4] = X_1, [X_3, X_4] = X_2, [X_1, X_5] = \frac{2}{3}X_1,$ $[X_2, X_5] = -\frac{1}{3}X_2, [X_3, X_5] = -\frac{4}{3}X_3, [X_4, X_5] = X_4$	yes

Table A.7: 5-dimensional indecomposable unimodular algebras with nilradical $3\mathfrak{g}_1$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{5.33}^{-1,-1}$	$[X_1, X_4] = X_1, [X_3, X_4] = -X_3,$ $[X_2, X_5] = X_2, [X_3, X_5] = -X_3$	yes
$\mathfrak{g}_{5.35}^{-2,0}$	$[X_1, X_4] = -2X_1, [X_2, X_4] = X_2, [X_3, X_4] = X_3,$ $[X_2, X_5] = -X_3, [X_3, X_5] = X_2$	no

There are ten classes of decomposable nilpotent Lie algebras in dimension six: $6\mathfrak{g}_1, \mathfrak{g}_{3.1} \oplus 3\mathfrak{g}_1, 2\mathfrak{g}_{3.1}, \mathfrak{g}_{4.1} \oplus 2\mathfrak{g}_1$ and $\mathfrak{g}_{5.i} \oplus \mathfrak{g}_1$ for $i \in \{1, \dots, 6\}$.

Tables A.8 and A.9 contain the six-dimensional indecomposable nilpotent real Lie algebras. They base on Morozov's classification in [54], where nilpotent algebras over a field of characteristic zero are determined. Note that over \mathbb{R} , there is only one isomorphism class of Morozov's indecomposable type 5 resp. type 10 and type 14 resp. 18 splits into two non-isomorphic ones.

Table A.8: 6-dimensional indecomposable nilpotent algebras

	$[X_i, X_j]$
$\mathfrak{g}_{6.N1}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_5] = X_6$
$\mathfrak{g}_{6.N2}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_1, X_5] = X_6$
$\mathfrak{g}_{6.N3}$	$[X_1, X_2] = X_6, [X_1, X_3] = X_4, [X_2, X_3] = X_5$
$\mathfrak{g}_{6.N4}$	$[X_1, X_2] = X_5, [X_1, X_3] = X_6, [X_2, X_4] = X_6$
$\mathfrak{g}_{6.N5}$	$[X_1, X_3] = X_5, [X_1, X_4] = X_6, [X_2, X_3] = -X_6, [X_2, X_4] = X_5$
$\mathfrak{g}_{6.N6}$	$[X_1, X_2] = X_6, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_5$
$\mathfrak{g}_{6.N7}$	$[X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_6$
$\mathfrak{g}_{6.N8}$	$[X_1, X_2] = X_3 + X_5, [X_1, X_3] = X_4, [X_2, X_5] = X_6$
$\mathfrak{g}_{6.N9}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_5] = X_6, [X_2, X_3] = X_5$
$\mathfrak{g}_{6.N10}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_5, [X_1, X_4] = X_6,$ $[X_2, X_3] = -X_6, [X_2, X_4] = X_5$
$\mathfrak{g}_{6.N11}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_6$
$\mathfrak{g}_{6.N12}$	$[X_1, X_3] = X_4, [X_1, X_4] = X_6, [X_2, X_5] = X_6$
$\mathfrak{g}_{6.N13}$	$[X_1, X_2] = X_5, [X_1, X_3] = X_4, [X_1, X_4] = X_6, [X_2, X_5] = X_6$
$\mathfrak{g}_{6.N14}^{\pm 1}$	$[X_1, X_3] = X_4, [X_1, X_4] = X_6, [X_2, X_3] = X_5, [X_2, X_5] = \pm X_6$
$\mathfrak{g}_{6.N15}$	$[X_1, X_2] = X_3 + X_5, [X_1, X_3] = X_4, [X_1, X_4] = X_6, [X_2, X_5] = X_6$

Table A.9: 6-dimensional indecomposable nilpotent algebras (continued)

	$[X_i, X_j]$
$\mathfrak{g}_{6.N16}$	$[X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_1, X_5] = X_6,$ $[X_2, X_3] = X_5, [X_2, X_4] = X_6$
$\mathfrak{g}_{6.N17}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_6, [X_2, X_5] = X_6$
$\mathfrak{g}_{6.N18}^{\pm 1}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_6,$ $[X_2, X_3] = X_5, [X_2, X_5] = \pm X_6$
$\mathfrak{g}_{6.N19}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5,$ $[X_1, X_5] = X_6, [X_2, X_3] = X_6$
$\mathfrak{g}_{6.N20}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5,$ $[X_1, X_5] = X_6, [X_2, X_3] = X_5, [X_2, X_4] = X_6$
$\mathfrak{g}_{6.N21}$	$[X_1, X_2] = X_3, [X_1, X_5] = X_6, [X_2, X_3] = X_4,$ $[X_2, X_4] = X_5, [X_3, X_4] = X_6$
$\mathfrak{g}_{6.N22}$	$[X_1, X_2] = X_3, [X_1, X_3] = X_5, [X_1, X_5] = X_6,$ $[X_2, X_3] = X_4, [X_2, X_4] = X_5, [X_3, X_4] = X_6$

Mubarakzjanov's list in [58] contains 99 classes of six-dimensional indecomposable almost nilpotent Lie algebras, namely $\mathfrak{g}_{6.1}, \dots, \mathfrak{g}_{6.99}$.

As first remarked by Turkowski, there is one algebra missing. The complete (and partly corrected) list can be found in the article [11] of Campoamor-Stursberg¹, where the missing algebra is denoted by $\mathfrak{g}_{6.92}^*$.

We list the unimodular among this 100 algebras in Tables A.10 to A.23 (where some minor misprints have been corrected). Note that there is no table with Lie algebras with nilradical $\mathfrak{g}_{5.6}$ since the only such algebra is not unimodular.

Table A.10: 6-dimensional indecomposable unimodular almost abelian algebras

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.1}^{a,b,c,d}$	$[X_1, X_6] = X_1, [X_2, X_6] = aX_2, [X_3, X_6] = bX_3,$ $[X_4, X_6] = cX_4, [X_5, X_6] = dX_5,$ $0 < d \leq c \leq b \leq a \leq 1, a + b + c + d = -1$	yes

¹The author wishes to express his gratitude to R. Campoamor-Stursberg for providing him with copies of [11] and [58].

Table A.11: 6-dimensional indecomposable unimodular almost abelian algebras (continued)

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.2}^{a,c,d}$	$[X_1, X_6] = aX_1, [X_2, X_6] = X_1 + aX_2, [X_3, X_6] = X_3,$ $[X_4, X_6] = cX_4, [X_5, X_6] = dX_5,$ $0 < d \leq c \leq 1, 2a + c + d = -1$	yes
$\mathfrak{g}_{6.3}^{-\frac{d+1}{3},d}$	$[X_1, X_6] = -\frac{d+1}{3}X_1, [X_2, X_6] = X_1 - \frac{d+1}{3}X_2,$ $[X_3, X_6] = X_2 - \frac{d+1}{3}X_3, [X_4, X_6] = X_4, [X_5, X_6] = dX_5,$ $0 < d \leq 1,$	yes
$\mathfrak{g}_{6.4}^{-\frac{1}{4}}$	$[X_1, X_6] = -\frac{1}{4}X_1, [X_2, X_6] = X_1 - \frac{1}{4}X_2,$ $[X_3, X_6] = X_2 - \frac{1}{4}X_3, [X_4, X_6] = X_3 - \frac{1}{4}X_4, [X_5, X_6] = X_5$	yes
$\mathfrak{g}_{6.6}^{a,b}$	$[X_1, X_6] = X_1, [X_2, X_6] = aX_2, [X_3, X_6] = X_2 + aX_3,$ $[X_4, X_6] = bX_4, [X_5, X_6] = X_4 + bX_5, a \leq b, a + b = -\frac{1}{2}$	yes
$\mathfrak{g}_{6.7}^{a,-\frac{2}{3}a}$	$[X_1, X_6] = aX_1, [X_2, X_6] = X_1 + aX_2, [X_3, X_6] = X_2 + aX_3,$ $[X_4, X_6] = -\frac{3}{2}aX_4, [X_5, X_6] = X_4 - \frac{3}{2}aX_5, a \neq 0$	yes
$\mathfrak{g}_{6.8}^{a,b,c,p}$	$[X_1, X_6] = aX_1, [X_2, X_6] = bX_2, [X_3, X_6] = cX_3,$ $[X_4, X_6] = pX_4 - X_5, [X_5, X_6] = X_4 + pX_5,$ $0 < c \leq b \leq a , a + b + c + 2p = 0$	no
$\mathfrak{g}_{6.9}^{a,b,p}$	$[X_1, X_6] = aX_1, [X_2, X_6] = bX_2, [X_3, X_6] = X_2 + bX_3,$ $[X_4, X_6] = pX_4 - X_5, [X_5, X_6] = X_4 + pX_5,$ $a \neq 0, a + 2b + 2p = 0$	no
$\mathfrak{g}_{6.10}^{a,-\frac{3}{2}a}$	$[X_1, X_6] = aX_1, [X_2, X_6] = X_1 + aX_2, [X_3, X_6] = X_2 + aX_3,$ $[X_4, X_6] = -\frac{3}{2}aX_4 - X_5, [X_5, X_6] = X_4 - \frac{3}{2}aX_5$	no
$\mathfrak{g}_{6.11}^{a,p,q,s}$	$[X_1, X_6] = aX_1, [X_2, X_6] = pX_2 - X_3, [X_3, X_6] = X_2 + pX_3,$ $[X_4, X_6] = qX_4 - sX_5, [X_5, X_6] = sX_4 + qX_5,$ $as \neq 0, a + 2p + 2q = 0$	no
$\mathfrak{g}_{6.12}^{-4p,p}$	$[X_1, X_6] = -4pX_1, [X_2, X_6] = pX_2 - X_3,$ $[X_3, X_6] = X_2 + pX_3, [X_4, X_6] = X_2 + pX_4 - X_5,$ $[X_5, X_6] = X_3 + X_4 + pX_5, p \neq 0$	no

Table A.12: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{3.1} \oplus 2\mathfrak{g}_1$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.13}^{a,b,h}$	$[X_2, X_3] = X_1, [X_1, X_6] = (a+b)X_1, [X_2, X_6] = aX_2,$ $[X_3, X_6] = bX_3, [X_4, X_6] = X_4, [X_5, X_6] = hX_5,$ $a \neq 0, 2a + 2b + h = -1$	yes
$\mathfrak{g}_{6.14}^{a,b}$	$[X_2, X_3] = X_1, [X_1, X_6] = (a+b)X_1, [X_2, X_6] = aX_2,$ $[X_3, X_6] = bX_3, [X_4, X_6] = X_4, [X_5, X_6] = X_1 + (a+b)X_5,$ $a \neq 0, a + b = -\frac{1}{3}$	yes

Table A.13: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{3.1} \oplus 2\mathfrak{g}_1$ (continued)

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.15}^{-1}$	$[X_2, X_3] = X_1, [X_2, X_6] = X_2 + X_4,$ $[X_3, X_6] = -X_3 + X_5, [X_4, X_6] = X_4, [X_5, X_6] = -X_5,$	yes
$\mathfrak{g}_{6.17}^{-\frac{1}{2}, 0}$	$[X_2, X_3] = X_1, [X_1, X_6] = -\frac{1}{2}X_1, [X_2, X_6] = -\frac{1}{2}X_2,$ $[X_3, X_6] = X_4, [X_5, X_6] = X_5,$	yes
$\mathfrak{g}_{6.18}^{a, -2a-3}$	$[X_2, X_3] = X_1, [X_1, X_6] = (1+a)X_1, [X_2, X_6] = aX_2,$ $[X_3, X_6] = X_3 + X_4, [X_4, X_6] = X_4,$ $[X_5, X_6] = -(2a+3)X_5, a \neq -\frac{3}{2}$	yes
$\mathfrak{g}_{6.19}^{-\frac{4}{3}}$	$[X_2, X_3] = X_1, [X_1, X_6] = -\frac{1}{3}X_1, [X_2, X_6] = -\frac{4}{3}X_2,$ $[X_3, X_6] = X_3 + X_4, [X_4, X_6] = X_4, [X_5, X_6] = X_1 - \frac{1}{3}X_5$	yes
$\mathfrak{g}_{6.20}^{-3}$	$[X_2, X_3] = X_1, [X_1, X_6] = X_1, [X_3, X_6] = X_3 + X_4,$ $[X_4, X_6] = X_1 + X_4, [X_5, X_6] = -3X_5$	yes
$\mathfrak{g}_{6.21}^a$	$[X_2, X_3] = X_1, [X_1, X_6] = 2aX_1, [X_2, X_6] = aX_2 + X_3,$ $[X_3, X_6] = aX_3, [X_4, X_6] = X_4, [X_5, X_6] = -(4a+1)X_5,$ $a \neq -\frac{1}{4}$	yes
$\mathfrak{g}_{6.22}^{-\frac{1}{6}}$	$[X_2, X_3] = X_1, [X_1, X_6] = -\frac{1}{3}X_1, [X_2, X_6] = -\frac{1}{6}X_2 + X_3,$ $[X_3, X_6] = -\frac{1}{6}X_3, [X_4, X_6] = X_4, [X_5, X_6] = X_1 - \frac{1}{3}X_5$	yes
$\mathfrak{g}_{6.23}^{a, -7a, \varepsilon}$	$[X_2, X_3] = X_1, [X_1, X_6] = 2aX_1, [X_2, X_6] = aX_2 + X_3,$ $[X_3, X_6] = aX_3 + X_4, [X_4, X_6] = aX_4,$ $[X_5, X_6] = \varepsilon X_1 - 5aX_5, \varepsilon a = 0$	yes
$\mathfrak{g}_{6.25}^{b, -1-b}$	$[X_2, X_3] = X_1, [X_1, X_6] = -bX_1,$ $[X_2, X_6] = X_2, [X_3, X_6] = -(1+b)X_3,$ $[X_4, X_6] = bX_4 + X_5, [X_5, X_6] = bX_5$	yes
$\mathfrak{g}_{6.26}^{-1}$	$[X_2, X_3] = X_1, [X_2, X_6] = X_2, [X_3, X_6] = -X_3$ $[X_4, X_6] = X_5, [X_5, X_6] = X_1$	yes
$\mathfrak{g}_{6.27}^{-2b, b, 0}$	$[X_2, X_3] = X_1, [X_1, X_6] = -bX_1, [X_2, X_6] = -2bX_2,$ $[X_3, X_6] = bX_3 + X_4, [X_4, X_6] = bX_4 + X_5,$ $[X_5, X_6] = bX_5, b \neq 0$	yes
$\mathfrak{g}_{6.28}^{-2}$	$[X_2, X_3] = X_1, [X_1, X_6] = 2X_1, [X_2, X_6] = X_2 + X_3,$ $[X_3, X_6] = X_3, [X_4, X_6] = -2X_4 + X_5, [X_5, X_6] = -2X_5$	yes
$\mathfrak{g}_{6.29}^{-2b, b, \varepsilon}$	$[X_2, X_3] = X_1, [X_1, X_6] = -bX_1, [X_2, X_6] = -2bX_2,$ $[X_3, X_6] = bX_3 + X_4, [X_4, X_6] = bX_4 + X_5,$ $[X_5, X_6] = \varepsilon X_1 + bX_5, \varepsilon b = 0 (?)$	yes
$\mathfrak{g}_{6.32}^{a, -6a-h, h, \varepsilon}$	$[X_2, X_3] = X_1, [X_1, X_6] = 2aX_1, [X_2, X_6] = aX_2 + X_3,$ $[X_3, X_6] = -X_2 + aX_3, [X_4, X_6] = \varepsilon X_1 + (2a+h)X_4,$ $[X_5, X_6] = -(6a+h)X_5, a > -\frac{1}{4}h, \varepsilon h = 0$	no

Table A.14: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{3.1} \oplus 2\mathfrak{g}_1$ (continued)

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.33}^{a,-6a}$	$[X_2, X_3] = X_1, [X_1, X_6] = 2aX_1, [X_2, X_6] = aX_2 + X_3,$ $[X_3, X_6] = -X_2 + aX_3, [X_4, X_6] = -6aX_4,$ $[X_5, X_6] = X_1 + 2aX_5, a \geq 0$	no
$\mathfrak{g}_{6.34}^{a,-4a,\varepsilon}$	$[X_2, X_3] = X_1, [X_1, X_6] = 2aX_1, [X_2, X_6] = aX_2 + X_3,$ $[X_3, X_6] = -X_2 + aX_3, [X_4, X_6] = -2aX_4,$ $[X_5, X_6] = \varepsilon X_1 - 2aX_5, \varepsilon a = 0$	no
$\mathfrak{g}_{6.35}^{a,b,c}$	$[X_2, X_3] = X_1, [X_1, X_6] = (a+b)X_1, [X_2, X_6] = aX_2,$ $[X_3, X_6] = bX_3, [X_4, X_6] = cX_4 + X_5,$ $[X_5, X_6] = -X_4 + cX_5, a+b+c=0, a^2+b^2 \neq 0$	no
$\mathfrak{g}_{6.36}^{a,-2a}$	$[X_2, X_3] = X_1, [X_1, X_6] = 2aX_1, [X_2, X_6] = aX_2 + X_3,$ $[X_3, X_6] = aX_3, [X_4, X_6] = -2aX_4 + X_5,$ $[X_5, X_6] = -X_4 - 2aX_5$	no
$\mathfrak{g}_{6.37}^{-a,-2a,s}$	$[X_2, X_3] = X_1, [X_1, X_6] = 2aX_1, [X_2, X_6] = aX_2 + X_3,$ $[X_3, X_6] = -X_2 + aX_3, [X_4, X_6] = -2aX_4 + sX_5,$ $[X_5, X_6] = -sX_4 - 2aX_5, s \neq 0$	no
$\mathfrak{g}_{6.38}^0$	$[X_2, X_3] = X_1, [X_2, X_6] = X_3 + X_4,$ $[X_3, X_6] = -X_2 + X_5, [X_4, X_6] = X_5, [X_5, X_6] = -X_4$	no

Table A.15: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{4.1} \oplus \mathfrak{g}_1$

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.39}^{-4-3h,h}$	$[X_1, X_5] = X_2, [X_4, X_5] = X_1, [X_1, X_6] = (1+h)X_1,$ $[X_2, X_6] = (2+h)X_2, [X_3, X_6] = -(4+3h)X_3,$ $[X_4, X_6] = hX_4, [X_5, X_6] = X_5, h \neq -\frac{4}{3}$	yes
$\mathfrak{g}_{6.40}^{-\frac{3}{2}}$	$[X_1, X_5] = X_2, [X_4, X_5] = X_1,$ $[X_1, X_6] = -\frac{1}{2}X_1, [X_2, X_6] = \frac{1}{2}X_2,$ $[X_3, X_6] = X_2 + \frac{1}{2}X_3, [X_4, X_6] = -\frac{3}{2}X_4, [X_5, X_6] = X_5$	yes
$\mathfrak{g}_{6.41}^{-1}$	$[X_1, X_5] = X_2, [X_4, X_5] = X_1,$ $[X_2, X_6] = X_2, [X_3, X_6] = -X_3,$ $[X_4, X_6] = X_3 - X_4, [X_5, X_6] = X_5$	yes

Table A.16: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{4.1} \oplus \mathfrak{g}_1$ (continued)

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.42}^{-\frac{5}{3}}$	$[X_1, X_5] = X_2, [X_4, X_5] = X_1,$ $[X_1, X_6] = -\frac{2}{3}X_1, [X_2, X_6] = \frac{1}{3}X_2, [X_3, X_6] = X_3,$ $[X_4, X_6] = -\frac{5}{3}X_4, [X_5, X_6] = X_3 + X_5$	yes
$\mathfrak{g}_{6.44}^{-7}$	$[X_1, X_5] = X_2, [X_4, X_5] = X_1,$ $[X_1, X_6] = 2X_1, [X_2, X_6] = 3X_2, [X_3, X_6] = -7X_3,$ $[X_4, X_6] = X_4, [X_5, X_6] = X_4 + X_5$	yes
$\mathfrak{g}_{6.47}^{-3,\varepsilon}$	$[X_1, X_5] = X_2, [X_4, X_5] = X_1,$ $[X_1, X_6] = X_1, [X_2, X_6] = X_2, [X_3, X_6] = -3X_3,$ $[X_4, X_6] = \varepsilon X_2 + X_4, \varepsilon \in \{0, \pm 1\}$	yes

Table A.17: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{5.1}$

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.54}^{2(1+l),l}$	$[X_3, X_5] = X_1, [X_4, X_5] = X_2,$ $[X_1, X_6] = X_1, [X_2, X_6] = lX_2, [X_3, X_6] = (-1 - 2l)X_3,$ $[X_4, X_6] = (-2 - l)X_4, [X_5, X_6] = 2(1 + l)X_5$	yes
$\mathfrak{g}_{6.55}^{-4}$	$[X_3, X_5] = X_1, [X_4, X_5] = X_2,$ $[X_1, X_6] = X_1, [X_2, X_6] = -3X_2, [X_3, X_6] = 4X_3,$ $[X_4, X_6] = X_1 + X_4, [X_5, X_6] = -3X_5$	yes
$\mathfrak{g}_{6.56}^{\frac{4}{3}}$	$[X_3, X_5] = X_1, [X_4, X_5] = X_2,$ $[X_1, X_6] = X_1, [X_2, X_6] = -\frac{1}{3}X_2, [X_3, X_6] = X_2 - \frac{1}{3}X_3,$ $[X_4, X_6] = -\frac{5}{3}X_4, [X_5, X_6] = \frac{4}{3}X_5$	yes
$\mathfrak{g}_{6.57}^{-\frac{2}{3}}$	$[X_3, X_5] = X_1, [X_4, X_5] = X_2,$ $[X_1, X_6] = X_1, [X_2, X_6] = -\frac{4}{3}X_2, [X_3, X_6] = \frac{5}{3}X_3,$ $[X_4, X_6] = -\frac{2}{3}X_4, [X_5, X_6] = X_4 - \frac{2}{3}X_5$	yes
$\mathfrak{g}_{6.61}^{-\frac{3}{4}}$	$[X_3, X_5] = X_1, [X_4, X_5] = X_2,$ $[X_1, X_6] = 2X_1, [X_2, X_6] = -\frac{3}{2}X_2, [X_3, X_6] = X_3,$ $[X_4, X_6] = -\frac{5}{2}X_4, [X_5, X_6] = X_3 + X_5$	yes
$\mathfrak{g}_{6.63}^{-1}$	$[X_3, X_5] = X_1, [X_4, X_5] = X_2,$ $[X_1, X_6] = X_1, [X_2, X_6] = -X_2, [X_3, X_6] = X_3,$ $[X_4, X_6] = X_2 - X_4$	yes
$\mathfrak{g}_{6.65}^{4l,l}$	$[X_3, X_5] = X_1, [X_4, X_5] = X_2,$ $[X_1, X_6] = lX_1 + X_2, [X_2, X_6] = lX_2, [X_3, X_6] = -3lX_3 + X_4,$ $[X_4, X_6] = -3lX_4, [X_5, X_6] = 4lX_5$	yes

Table A.18: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{5.1}$ (continued)

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.70}^{4p,p}$	$[X_3, X_5] = X_1, [X_4, X_5] = X_2,$ $[X_1, X_6] = pX_1 + X_2, [X_2, X_6] = -X_1 + pX_2,$ $[X_3, X_6] = -3pX_3 + X_4, [X_4, X_6] = -X_3 - 3pX_4,$ $[X_5, X_6] = 4pX_5$	no

Table A.19: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{5.2}$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.71}^{-\frac{7}{4}}$	$[X_2, X_5] = X_1, [X_3, X_5] = X_2, [X_4, X_5] = X_3,$ $[X_1, X_6] = \frac{5}{4}X_1, [X_2, X_6] = \frac{1}{4}X_2, [X_3, X_6] = -\frac{3}{4}X_3,$ $[X_4, X_6] = -\frac{7}{4}X_4, [X_5, X_6] = X_5$	yes

Table A.20: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{5.3}$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.76}^{-1}$	$[X_2, X_4] = X_3, [X_2, X_5] = X_1, [X_4, X_5] = X_2,$ $[X_1, X_6] = -X_1, [X_3, X_6] = X_3,$ $[X_4, X_6] = X_4, [X_5, X_6] = -X_5$	yes
$\mathfrak{g}_{6.78}$	$[X_2, X_4] = X_3, [X_2, X_5] = X_1, [X_4, X_5] = X_2,$ $[X_1, X_6] = -X_1, [X_3, X_6] = X_3,$ $[X_4, X_6] = X_3 + X_4, [X_5, X_6] = -X_5$	yes

Table A.21: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{5.4}$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.83}^{0,l}$	$[X_2, X_4] = X_1, [X_3, X_5] = X_1,$ $[X_2, X_6] = lX_2, [X_3, X_6] = lX_3,$ $[X_4, X_6] = -lX_4, [X_5, X_6] = -X_4 - lX_5$	yes
$\mathfrak{g}_{6.84}$	$[X_2, X_4] = X_1, [X_3, X_5] = X_1,$ $[X_2, X_6] = X_2, [X_4, X_6] = -X_4, [X_5, X_6] = X_3$	yes

Table A.22: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{5.4}$ (continued)

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.88}^{0, \mu_0, \nu_0}$	$[X_2, X_4] = X_1, [X_3, X_5] = X_1,$ $[X_2, X_6] = \mu_0 X_2 + \nu_0 X_3, [X_3, X_6] = -\nu_0 X_2 + \mu_0 X_3,$ $[X_4, X_6] = -\mu_0 X_4 + \nu_0 X_5, [X_5, X_6] = -\nu_0 X_4 - \mu_0 X_5$	cpl. solv. \Updownarrow $\nu_0 = 0$
$\mathfrak{g}_{6.89}^{0, \nu_0, s}$	$[X_2, X_4] = X_1, [X_3, X_5] = X_1,$ $[X_2, X_6] = sX_2, [X_3, X_6] = \nu_0 X_5,$ $[X_4, X_6] = -sX_4, [X_5, X_6] = -\nu_0 X_3$	cpl. solv. \Updownarrow $\nu_0 = 0$
$\mathfrak{g}_{6.90}^{0, \nu_0}$	$[X_2, X_4] = X_1, [X_3, X_5] = X_1,$ $[X_2, X_6] = X_4, [X_3, X_6] = \nu_0 X_5,$ $[X_4, X_6] = X_2, [X_5, X_6] = -\nu_0 X_3, \nu_0 \neq 1$	cpl. solv. \Updownarrow $\nu_0 = 0$
$\mathfrak{g}_{6.91}$	$[X_2, X_4] = X_1, [X_3, X_5] = X_1,$ $[X_2, X_6] = X_4, [X_3, X_6] = X_5,$ $[X_4, X_6] = X_2, [X_5, X_6] = -X_3$	no
$\mathfrak{g}_{6.92}^{0, \mu_0, \nu_0}$	$[X_2, X_4] = X_1, [X_3, X_5] = X_1,$ $[X_2, X_6] = \nu_0 X_3, [X_3, X_6] = -\mu_0 X_2,$ $[X_4, X_6] = \mu_0 X_5, [X_5, X_6] = -\nu_0 X_4$	no
$\mathfrak{g}_{6.92}^0$	$[X_2, X_4] = X_1, [X_3, X_5] = X_1,$ $[X_2, X_6] = X_4, [X_3, X_6] = X_5,$ $[X_4, X_6] = -X_2, [X_5, X_6] = -X_3$	no
$\mathfrak{g}_{6.93}^{0, \nu_0}$	$[X_2, X_4] = X_1, [X_3, X_5] = X_1,$ $[X_2, X_6] = X_4 + \nu_0 X_5, [X_3, X_6] = \nu_0 X_4,$ $[X_4, X_6] = X_2 - \nu_0 X_3, [X_5, X_6] = -\nu_0 X_2$	cpl. solv. \Updownarrow $ \nu_0 \leq \frac{1}{2}$

Table A.23: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{5.5}$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.94}^{-2}$	$[X_3, X_4] = X_1, [X_2, X_5] = X_1, [X_3, X_5] = X_2,$ $[X_2, X_6] = -X_2, [X_3, X_6] = -2X_3,$ $[X_4, X_6] = 2X_4, [X_5, X_6] = X_5$	yes

The six-dimensional solvable Lie algebras with four-dimensional nilradical were classified by Turkowski in [75]. We list the unimodular among them in Tables A.24 – A.26. Note that there is no table with Lie algebras with nilradical $\mathfrak{g}_{4.1}$ since the only such algebra is not unimodular.

The equations for the twenty-fifth algebra in Turkowski's list contain a minor misprint that we have corrected here.

Table A.24: 6-dimensional indecomposable unimodular algebras with nilradical $4\mathfrak{g}_1$

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.101}^{a,b,c,d}$	$[X_5, X_1] = aX_1, [X_5, X_2] = cX_2, [X_5, X_4] = X_4,$ $[X_6, X_1] = bX_1, [X_6, X_2] = dX_2, [X_6, X_3] = X_3,$ $a + c = -1, b + d = -1, ab \neq 0, c^2 + d^2 \neq 0$	yes
$\mathfrak{g}_{6.102}^{-1,b,-2-b}$	$[X_5, X_1] = -X_1, [X_5, X_2] = X_2, [X_5, X_3] = X_4,$ $[X_6, X_1] = bX_1, [X_6, X_2] = (-2 - b)X_2,$ $[X_6, X_3] = X_3, [X_6, X_4] = X_4$	yes
$\mathfrak{g}_{6.105}^{-2,-1}$	$[X_5, X_1] = -2X_1, [X_5, X_3] = X_3 + X_4,$ $[X_5, X_4] = X_4, [X_6, X_1] = -X_1, [X_6, X_2] = X_2$	yes
$\mathfrak{g}_{6.107}^{-1,b,0}$	$[X_5, X_1] = -X_1, [X_5, X_2] = -X_2, [X_5, X_3] = X_3 + X_4,$ $[X_5, X_4] = X_4, [X_6, X_1] = X_2, [X_6, X_2] = -X_1$	no
$\mathfrak{g}_{6.113}^{a,b,-a,d}$	$[X_5, X_1] = aX_1, [X_5, X_2] = -aX_2, [X_5, X_3] = X_4,$ $[X_6, X_1] = bX_1, [X_6, X_2] = dX_2, [X_6, X_3] = X_3,$ $[X_6, X_4] = X_4, a^2 + b^2 \neq 0, a^2 + d^2 \neq 0, b + d = -2$	no
$\mathfrak{g}_{6.114}^{a,-1,-\frac{a}{2}}$	$[X_5, X_1] = aX_1, [X_5, X_3] = -\frac{a}{2}X_3 + X_4,$ $[X_5, X_4] = -X_3 + \frac{a}{2}X_4, [X_6, X_1] = -X_1,$ $[X_6, X_2] = X_2, a \neq 0$	no
$\mathfrak{g}_{6.115}^{-1,b,c,-c}$	$[X_5, X_1] = X_1, [X_5, X_2] = X_2,$ $[X_5, X_3] = -X_3 + bX_4, [X_5, X_4] = -bX_3 - X_4,$ $[X_6, X_1] = cX_1 + X_2, [X_6, X_2] = -X_1 + cX_2,$ $[X_6, X_3] = -cX_3, [X_6, X_4] = -cX_4, b \neq 0$	no
$\mathfrak{g}_{6.116}^{0,-1}$	$[X_5, X_1] = X_2, [X_5, X_3] = X_4, [X_5, X_4] = -X_3,$ $[X_6, X_1] = X_1, [X_6, X_2] = X_2,$ $[X_6, X_3] = -X_3, [X_6, X_4] = -X_4$	no
$\mathfrak{g}_{6.118}^{0,b,-1}$	$[X_5, X_1] = X_2, [X_5, X_2] = -X_1, [X_5, X_3] = bX_4,$ $[X_5, X_4] = -bX_3, [X_6, X_1] = X_1, [X_6, X_2] = X_2,$ $[X_6, X_3] = -X_3, [X_6, X_4] = -X_4, b \neq 0$	no
$\mathfrak{g}_{6.120}^{-1,-1}$	$[X_5, X_2] = -X_2, [X_5, X_4] = X_4, [X_5, X_6] = X_1,$ $[X_6, X_2] = -X_1, [X_6, X_3] = X_3$	yes
$\mathfrak{g}_{6.125}^{0,-2}$	$[X_5, X_3] = X_4, [X_5, X_4] = -X_3, [X_5, X_6] = X_1,$ $[X_6, X_2] = -2X_2, [X_6, X_3] = X_3, [X_6, X_4] = X_4$	no

Table A.25: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.129}^{-2,-2}$	$[X_2, X_3] = X_1, [X_5, X_1] = X_1, [X_5, X_2] = X_2,$ $[X_5, X_4] = -2X_4, [X_6, X_1] = X_1,$ $[X_6, X_3] = X_3, [X_6, X_4] = -2X_4$	yes

Table A.26: 6-dimensional indecomposable unimodular algebras with nilradical $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ (continued)

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.135}^{0,-4}$	$[X_2, X_3] = X_1, [X_5, X_2] = X_3, [X_5, X_3] = -X_1,$ $[X_6, X_1] = 2X_1, [X_6, X_2] = X_2,$ $[X_6, X_3] = X_3, [X_6, X_4] = -4X_4$	no

In the introduction of [58], Mubarakzjanov quotes his own result that a six-dimensional solvable Lie algebra with three-dimensional nilradical is decomposable. Therefore, by Proposition 3.2.5, we have listed all unimodular indecomposable solvable Lie algebras of dimension six.

The first Betti numbers of the six-dimensional unimodular indecomposable Lie algebras are listed in Tables A.27 – A.29. The word “always” means that the certain value arises independent of the parameters on which the Lie algebra depends, but we suppose that the parameters are chosen such that Lie algebra is unimodular. The word “otherwise” in the tables means that this value arises for all parameters such that the Lie algebra is unimodular and the parameters are not mentioned in another column of the Lie algebra’s row.

Table A.27: $b_1(\mathfrak{g}_{6.i})$ for $\mathfrak{g}_{6.i}$ unimodular

i	$b_1 = 1$	$b_1 = 2$	$b_1 = 3$
1	always	-	-
2	$a \neq 0$	$a = 0$	-
3	$d \neq -1$	$d = -1$	-
4	always	-	-
6	$a, b \neq 0$	$a = -\frac{1}{2} \wedge b = 0$	-
7	always	-	-
8	always	-	-
9	$b \neq 0$	$b = 0$	-
10	$a \neq 0$	$a = 0$	-
11	always	-	-
12	always	-	-
13	$b \neq 0 \wedge h \neq 0$	otherwise	$a = -\frac{1}{2} \wedge b = h = 0$
14	otherwise	$a = -\frac{1}{3} \wedge b = 0$	-
15	always	-	-
17	-	always	-
18	$a \neq 0$	$a = 0$	-
19	always	-	-
20	-	always	-

Table A.28: $b_1(\mathfrak{g}_{6,i})$ for $\mathfrak{g}_{6,i}$ unimodular (continued)

i	$b_1 = 1$	$b_1 = 2$	$b_1 = 3$	$b_1 = 4$	$b_1 = 5$
21	$a \neq 0$	$a = 0$	-	-	-
22	always	-	-	-	-
23	$a \neq 0$	-	$a = 0$	-	-
25	$b \notin \{-1, 0\}$	$b \in \{-1, 0\}$	-	-	-
26	-	always	-	-	-
27	always	-	-	-	-
28	always	-	-	-	-
29	$b \neq 0$	-	$b = 0$	-	-
32	$h \notin \{-2a, -6a\}$	otherwise	-	-	-
33	$a \neq 0$	-	$a = 0$	-	-
34	$a \neq 0$	-	$a = 0$	-	-
35	$a, b \neq 0$	otherwise	-	-	-
36	$a \neq 0$	$a = 0$	-	-	-
37	always	-	-	-	-
38	always	-	-	-	-
39	$h \neq 0$	$h = 0$	-	-	-
40	always	-	-	-	-
41	always	-	-	-	-
42	always	-	-	-	-
44	always	-	-	-	-
47	-	always	-	-	-
54	$l \notin \{-2, -1, -\frac{1}{2}\}$	$l \in \{-2, -1, -\frac{1}{2}\}$	-	-	-
55	always	-	-	-	-
56	always	-	-	-	-
57	always	-	-	-	-
61	always	-	-	-	-
63	-	always	-	-	-
65	$l \neq 0$	-	$l = 0$	-	-
70	$p \neq 0$	$p = 0$	-	-	-
71	always	-	-	-	-
76	always	-	-	-	-
78	always	-	-	-	-
83	$l \neq 0$	-	-	$l = 0$	-
84	-	always	-	-	-
88	$\mu_0 \neq 0 \vee \nu_0 \neq 0$	-	-	-	$\mu_0 = \nu_0 = 0$
89	$\nu_0 \neq 0 \wedge s \neq 0$	-	otherwise	-	$\nu_0 = s = 0$
90	$\nu_0 \neq 0$	-	$\nu_0 = 0$	-	-

Table A.29: $b_1(\mathfrak{g}_{6.i})$ for $\mathfrak{g}_{6.i}$ unimodular (continued)

i	$b_1 = 1$	$b_1 = 2$	$b_1 = 3$	$b_1 = 4$	$b_1 = 5$
91	always	-	-	-	-
92	$\mu_0 \neq 0 \wedge \nu_0 \neq 0$	-	otherwise	-	$\mu_0 = \nu_0 = 0$
92*	always	-	-	-	-
93	$\nu_0 \neq 0$	-	$\nu_0 = 0$	-	-
94	always	-	-	-	-
101	-	always	-	-	-
102	-	always	-	-	-
105	-	always	-	-	-
107	-	always	-	-	-
113	-	always	-	-	-
114	-	always	-	-	-
115	-	always	-	-	-
116	-	always	-	-	-
118	-	always	-	-	-
120	-	always	-	-	-
125	-	always	-	-	-
129	-	always	-	-	-
135	-	always	-	-	-

Appendix B

Integer Polynomials

In this thesis, we often try to use necessary conditions for a matrix to be conjugated to an integer matrix. We state briefly the used results. Vice versa, we sometimes want to find integer matrices with given minimal polynomial. We also present a few constructions.

Let be $n \in \mathbb{N}_+$, \mathbb{K} a field and $A \in M(n, n; \mathbb{K})$. The *characteristic polynomial* of A is the monic polynomial

$$P_A(X) := \det(X \text{id} - A) \in \mathbb{K}[X],$$

and the *minimal polynomial* $m_A(X)$ is the unique monic divisor of lowest degree of $P_A(X)$ in $\mathbb{K}[X]$ such that $m_A(A) = 0$. (Note, by the theorem of Cayley-Hamilton, one has $P_A(A) = 0$.)

If two matrices are conjugated, then they have the same characteristic resp. minimal polynomials.

$\lambda \in \overline{\mathbb{K}}$ is called *root* of A if λ is a root of the characteristic polynomial, considered as polynomial in $\overline{\mathbb{K}}[X]$, where $\overline{\mathbb{K}}$ denotes the algebraic closure of \mathbb{K} .

The next proposition follows directly from [49, Corollaries XIV.2.2, XIV.2.3].

Proposition B.1. *Let $n \in \mathbb{N}_+$. If $A \in M(n, n; \mathbb{C})$ and $B \in M(n, n; \mathbb{Q})$ are conjugated via an element of $\text{GL}(n, \mathbb{C})$, then holds $P_A(X) = P_B(X) \in \mathbb{Q}[X]$, $m_A(X) = m_B(X) \in \mathbb{Q}[X]$ and $m_A(X)$ divides $P_A(X)$ in $\mathbb{Q}[X]$. \square*

Proposition B.2. *If $P(X) \in \mathbb{Z}[X]$, $m(X) \in \mathbb{Q}[X]$ are monic polynomials and $m(X)$ divides $P(X)$ in $\mathbb{Q}[X]$, then holds $m(X) \in \mathbb{Z}[X]$.*

Proof. Let $P(X), m(X)$ be as in the proposition and $f(X) \in \mathbb{Q}[X]$ non-constant with $P(X) = f(X)m(X)$. There exist $k, l \in \mathbb{Z} \setminus \{0\}$ such that

$$k f(X) = \sum_i a_i X^i, \quad l m(X) = \sum_j b_j X^j \in \mathbb{Z}[X]$$

are primitive. (An integer polynomial is called *primitive* if its coefficients are relatively prime.) We have

$$klP(X) = \left(\sum_i a_i X^i\right)\left(\sum_j b_j X^j\right)$$

and claim $kl = \pm 1$.

Otherwise, there is a prime $p \in \mathbb{N}$ that divides kl . Since the coefficients of $kf(X)$ resp. $lm(X)$ are relatively prime, there are minimal $i_0, j_0 \in \mathbb{N}$ such that p does not divide a_{i_0} resp. b_{j_0} .

The coefficient of $X^{i_0+j_0}$ of $klf(X)m(X)$ is

$$a_{i_0}b_{j_0} + a_{i_0-1}b_{j_0+1} + a_{i_0+1}b_{j_0-1} + \dots$$

and p divides each summand except the first. But since $p \mid kl$, p divides the whole sum. This is a contradiction. \square

Theorem B.3. *Let $n \in \mathbb{N}_+$ and $A \in M(n, n; \mathbb{C})$ be conjugated to an integer matrix. Then holds $P_A(X), m_A(X) \in \mathbb{Z}[X]$.*

Proof. This follows from the preceding two propositions. \square

Lemma B.4 ([40, Lemma 2.2]). *Let $P(X) = X^3 - kX^2 + lX - 1 \in \mathbb{Z}[X]$.*

Then P has a double root $X_0 \in \mathbb{R}$ if and only if $X_0 = 1$ or $X_0 = -1$ for which $P(X) = X^3 - 3X^2 + 3X - 1$ or $P(X) = X^3 + X^2 - X - 1$ respectively. \square

Proposition B.5 ([38, Proposition 5]). *Let $\lambda_i \in \mathbb{R}_+$ with $\lambda_i + \frac{1}{\lambda_i} = m_i \in \mathbb{N}_+$ and $m_i > 2$ for $i \in \{1, 2\}$.*

Then there exists no element in $\mathrm{SL}(3, \mathbb{Z})$ with roots $\lambda_1, \lambda_2, \frac{1}{\lambda_1\lambda_2}$. \square

Proposition B.6. *Let $P(X) = X^4 - mX^3 + pX^2 - nX + 1 \in \mathbb{Z}[X]$.*

Then P has a root with multiplicity > 1 if and only if the zero set of P equals $\{1, 1, a, a^{-1}\}$, $\{-1, -1, a, a^{-1}\}$, $\{a, a^{-1}, a, a^{-1}\}$ or $\{a, -a^{-1}, a, -a^{-1}\}$ for fixed $a \in \mathbb{C}$.

Proof. The most part of the proof was done by Harshavardhan in the proof of [38, Proposition 2].

We set $S := m^2 + n^2$ and $T := mn$ and get the discriminant D of $P(X)$ as

$$\begin{aligned} D &= 16p^4 - 4Sp^3 + (T^2 - 80T - 128)p^2 + 18S(T + 8)p \\ &\quad + 256 - 192T + 48T^2 - 4T^3 - 27S^2. \end{aligned} \quad (\text{B.1})$$

Note that $P(X)$ has a root of multiplicity > 1 if and only if $D = 0$. Solving $D = 0$ for S , we see

$$S = -\frac{2}{27}p^3 + \frac{1}{3}pT + \frac{8}{3}p \pm \frac{2}{27}\sqrt{(p^2 - 3T + 12)^3},$$

and since S and T are integers, there is $q \in \mathbb{Z}$ with

$$p^2 - 3T + 12 = q^2,$$

which implies

$$\begin{aligned} S &= 4p + \frac{1}{27}(p^3 - 3pq^2 \pm 2q^3) \\ T &= \frac{1}{3}(p^2 - q^2 + 12). \end{aligned} \tag{B.2}$$

We first consider the plus sign in equation (B.2). Then one has

$$\begin{aligned} (m+n)^2 &= S + 2T = \frac{1}{27}(p+2q+6)(p-q+6)^2, \\ (m-n)^2 &= S - 2T = \frac{1}{27}(p+2q-6)(p-q-6)^2, \end{aligned}$$

and this implies the existence of $k_i, l_i \in \mathbb{N}$, $i = 1, 2$, such that

$$\begin{aligned} 3k_1^2 &= (p+2q+6)k_2^2, \\ 3l_1^2 &= (p+2q-6)l_2^2. \end{aligned}$$

We shall show: $|m| = |n|$

[If $l_2 = 0$, the claim is proved. Therefore, we can assume $l_2 \neq 0$.

Case 1: $k_2 = 0$

Then holds $k_1 = 0$ and this means $S + 2T = 0$, i.e. $(m+n)^2 = 0$, so we have $m = -n$.

Case 2: $k_2 \neq 0$

We write $k := \frac{k_1}{k_2}$ and $l := \frac{l_1}{l_2}$. Then holds

$$\begin{aligned} 3k^2 &= p + 2q + 6 \in \mathbb{Z}, \\ 3l^2 &= p + 2q - 6 \in \mathbb{Z}, \end{aligned}$$

and $3(k^2 - l^2) = 12$. Therefore, we have $k^2 - l^2 = 4$, so $k^2 = 4$, $l^2 = 0$, i.e. $l_1 = 0$, $S - 2T = 0$ and $m = n$.]

Now, consider the minus sign in equation (B.2). Then one has

$$\begin{aligned} (m+n)^2 &= S + 2T = \frac{1}{27}(p-2q+6)(p+q+6)^2, \\ (m-n)^2 &= S - 2T = \frac{1}{27}(p-2q-6)(p+q-6)^2, \end{aligned}$$

and shows analogously as above $|m| = |n|$.

We have shown: If $P(X)$ has a multiple root, then holds $m = \pm n$.

If $m = n$, then one calculates the solutions of $D = 0$ in (B.1) as the following

$$(i) \quad p = -2 + 2m,$$

$$(ii) \quad p = -2 - 2m,$$

$$(iii) \quad p = 2 + \frac{m^2}{4},$$

and if $m = -n$, then the real solution of $D = 0$ in (B.1) is

$$(iv) \quad p = -2 + \frac{m^2}{4}.$$

Moreover, a short computation yields the zero set of $P(X)$ in the cases (i) – (iv) as $\{1, 1, a, a^{-1}\}$, $\{-1, -1, a, a^{-1}\}$, $\{a, a^{-1}, a, a^{-1}\}$, $\{a, -a^{-1}, a, -a^{-1}\}$, respectively. \square

Proposition B.7 ([1, Proposition 4.4.14]). *Let \mathbb{K} be a field and*

$$m(X) = X^n + a_{n-1}X^{n-1} + \dots + a_1X + a_0 \in \mathbb{K}[X]$$

a monic polynomial. Then

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -a_0 \\ 1 & 0 & \dots & 0 & 0 & -a_1 \\ 0 & 1 & \dots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \dots & 0 & 1 & -a_{n-1} \end{pmatrix} \text{ has minimal polynomial}$$

$m(X)$. \square

If one is willing to construct an integer matrix with given characteristic and minimal polynomial, one always can choose any matrix M which has the desired polynomials and try to find an invertible matrix T such that $T^{-1}MT$ has integer entries. Of course, this can be difficult. In the case of 4×4 - matrices we have the following easy construction.

Proposition B.8 ([38, Section 2.3.1]).

(i) *Let integers $m, n, p \in \mathbb{Z}$ be given.*

Choose $m_1, \dots, m_4 \in \mathbb{Z}$ such that $\sum_{i=1}^4 m_i = m$ and set

$$a := -m_1^2 p + m_1^3 m_2 + m_1^3 m_3 + m_1^3 m_4 + m_1 n - 1,$$

$$b := (-m_2 - m_1)p + m_1 m_2^2 + m_1 m_2 m_3 + m_1 m_2 m_4 + m_2^2 m_3 + m_2^2 m_4 \\ + m_1^2 m_2 + m_1^2 m_3 + m_1^2 m_4 + n,$$

$$c := m_1 m_2 + m_1 m_3 + m_1 m_4 + m_2 m_3 + m_2 m_4 + m_3 m_4 - p.$$

Then the matrix

$$\begin{pmatrix} m_1 & 0 & 0 & a \\ 1 & m_2 & 0 & b \\ 0 & 1 & m_3 & c \\ 0 & 0 & 1 & m_4 \end{pmatrix} \text{ has } X^4 - mX^3 + pX^2 - nX + 1$$

as characteristic polynomial.

(ii) Let $m \in 2\mathbb{Z}$ be an even integer. Then the matrix $\begin{pmatrix} \frac{m}{2} & 0 & -1 & 0 \\ 0 & \frac{m}{2} & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ has the characteristic polynomial $(X^2 - \frac{m}{2}X + 1)^2$, and $(X^2 - \frac{m}{2}X + 1)$ as minimal polynomial. \square

Appendix C

Group Extensions

Definition C.1. A *group extension* of a group Q by a group N is a short exact sequence of groups $\{1\} \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow \{1\}$.

If $s: Q \rightarrow G$ is a group homomorphism with $\pi \circ s = \text{id}$, then s is called *section*. In this case, we say that the extension is *split*.

Proposition C.2. A *group extension* $\{1\} \rightarrow N \rightarrow G \xrightarrow{\pi} F \rightarrow \{1\}$ of a free group F is *split*.

Proof. Let $\{x_i\}_i$ the set of generators of F . Choose $g_i \in G$ such that $\pi(g_i) = x_i$ and define a section $s: F \rightarrow G$ by $s(x_i) = g_i$. \square

Lemma C.3. Let $\{1\} \rightarrow A \xrightarrow{i} G \rightarrow Q \xrightarrow{\pi} \{1\}$ be an extension by an abelian group A . Then

$$q.a := i^{-1}(g_q \cdot i(a) \cdot g_q^{-1}) \text{ with any } g_q \in \pi^{-1}(\{q\})$$

defines a natural Q -module structure on A . \square

Lemma C.4. Let $\{1\} \rightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \rightarrow \{1\}$ be a split group extension with section $s: Q \rightarrow G$. Then

$$\mu: Q \times N \rightarrow N, \quad \mu((q, n)) = i^{-1}(s(q) \cdot i(n) \cdot s(q)^{-1}),$$

defines an action of Q on N by group automorphisms.

If N is abelian, the action coincides with the natural Q -module structure of N . \square

Recall the definition of the semidirect product $Q \ltimes_{\mu} N$ as set $Q \times N$ with group structure $(q_1, n_1)(q_2, n_2) = (q_1 q_2, \mu(q_2^{-1}, n_1) n_2)$.

Lemma C.5. Let N, Q be groups and $\mu: Q \times N \rightarrow N$ an action by group automorphisms.

Then $\{1\} \rightarrow N \xrightarrow{i_{\times}} Q \ltimes_{\mu} N \xrightarrow{\pi_{\times}} Q \rightarrow \{1\}$ is a split group extension, where $i_{\times}(n) = (n, e_Q)$ and $\pi_{\times}((q, n)) = q$. \square

Definition C.6. Let $\{1\} \rightarrow N \rightarrow G_k \rightarrow Q \rightarrow \{1\}$, $k \in \{1, 2\}$, be group extensions. They are called *equivalent*, if there is a group homomorphism $\varphi: G_1 \rightarrow G_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & N & \longrightarrow & G_1 & \longrightarrow & Q & \longrightarrow & \{1\} \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} & & \\ \{1\} & \longrightarrow & N & \longrightarrow & G_2 & \longrightarrow & Q & \longrightarrow & \{1\} \end{array}$$

Note, by the 5-Lemma, φ is necessary an isomorphism.

Proposition C.7. Let $\{1\} \rightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \rightarrow \{1\}$ be a split group extension with section $s: Q \rightarrow G$.

Then the extension is equivalent to the extension of N by Q which is given by the Lemmata C.4 and C.5.

Proof. Define $\varphi: Q \times_{\mu} N \rightarrow G$ by $f(q, n) = s(q) \cdot i(n)$. φ is a homomorphism since

$$\begin{aligned} \varphi((q_1, n_1)(q_2, n_2)) &= \varphi((q_1 q_2, \mu(q_2^{-1}, n_1) n_2)) = s(q_1 q_2) \cdot i(\mu(q_2^{-1}, n_1) n_2) \\ &= s(q_1 q_2) \cdot i(\mu(q_2^{-1}, n_1) n_2) \\ &= s(q_1) \cdot s(q_2) \cdot s(q_2)^{-1} \cdot i(n_1) \cdot s(q_2) \cdot i(n_2) \\ &= \varphi((q_1, s_1)) \cdot \varphi((q_2, n_2)). \end{aligned}$$

Further, φ satisfies the commutativity condition of the last lemma

$$\begin{aligned} \varphi(i_{\times}(n, e_Q)) &= s(e_Q) \cdot i(n) = i(n), \\ \pi(\varphi((q, n))) &= \pi(s(q) \cdot i(n)) = \pi(s(q)) \cdot \pi(i(n)) = q = \pi_{\times}((q, n)). \end{aligned}$$

Therefore, the proposition follows. \square

Definition C.8. Let $\{0\} \rightarrow A \xrightarrow{i} Q \times A \rightarrow Q \rightarrow \{1\}$ be a split extension by an abelian group A . Two sections $s_1, s_2: Q \rightarrow Q \times A$ are called *A-equivalent* if and only if there is an element $a \in A$ such that

$$s_1(q) = i(a) \cdot s_2(q) \cdot i(a)^{-1}$$

for all $q \in Q$.

Definition C.9. Let Q be a group and A a Q -module.

A 1-cocycle of Q with coefficients in A is a map $f: Q \rightarrow A$ with

$$\forall_{q_1, q_2 \in Q} f(q_1 q_2) = f(q_2) + q_2^{-1} \cdot f(q_1)$$

The set $Z^1(Q, A)$ of 1-cocycles is an abelian group with obvious group structure.

A 1-boundary of Q with coefficients in A is a map

$$f_a: Q \longrightarrow A, \quad f_a(q) = q^{-1}.a - a \text{ with fixed } a \in A.$$

The set $B^1(Q, A)$ of 1-boundaries forms a subgroup of $Z^1(Q, A)$.

The group $H^1(Q, A) := Z^1(Q, A)/B^1(Q, A)$ is called *first cohomology group of Q with coefficients in A* .

Theorem C.10 ([9, Proposition IV.2.3]). *Let $\{0\} \rightarrow A \rightarrow Q \times A \rightarrow Q \rightarrow \{1\}$ be a split extension by an abelian group A . Consider A with its natural Q -module structure.*

Then the elements of $H^1(Q, A)$ are in 1 – 1 correspondence with the set of A -conjugacy classes of sections $s: Q \rightarrow Q \times A$ via $[f] \mapsto [s(q) := (q, f(q))]$. \square

Definition C.11. Let Q be a group and A a Q -module.

A 2-cocycle of Q with coefficients in A is a map $f: Q \times Q \rightarrow A$ such that for all $q_1, q_2, q_3 \in Q$

$$q_3^{-1}.f(q_1, q_2) + f(q_1q_2, q_3) = f(q_1, q_2q_3) + f(q_2, q_3)$$

holds. The set $Z^2(Q, A)$ of 2-cocycles is an abelian group with obvious group structure.

A 2-boundary of Q with coefficients in A is a map

$$\begin{aligned} f_h: Q \times Q &\longrightarrow A \\ (q_1, q_2) &\longmapsto q_2^{-1}.h(q_1) - h(q_1q_2) + h(q_2) \end{aligned}$$

with a fixed map $h: Q \rightarrow A$.

The set $B^2(Q, A)$ of 2-boundaries forms a subgroup of $Z^2(Q, A)$.

The group $H^2(Q, A) := Z^2(Q, A)/B^2(Q, A)$ is called *second cohomology group of Q with coefficients in A* .

Theorem C.12 ([9, Theorem IV.3.12]). *Let Q be a group and A a Q -module.*

Then the elements of $H^2(Q, A)$ are in 1 – 1 correspondence with the set of equivalence classes of extensions of Q by A via $[f] \mapsto [E_f]$. E_f denotes the extension

$$\{0\} \longrightarrow A \xrightarrow{i} G_f \xrightarrow{\pi} Q \longrightarrow \{1\},$$

where $G_f := Q \times A$ with group structure given by

$$(q_1, a_1)(q_2, a_2) = (q_1q_2, q_2^{-1}.a_1 + a_2 + f(q_1, q_2)),$$

and inclusion $i(a) = (e_Q, a)$ as well as projection $\pi(q, a) = q$. \square

Bibliography

- [1] W. A. Adkins, St. H. Weintraub: *Algebra*, Springer (1992).
- [2] L. Auslander: *An exposition of the structure of solvmanifolds – part I*, Bull. Amer. Math. Soc. **79** (1973), no. 2, 227–261.
- [3] L. Auslander, L. Green, F. Hahn: *Flows on Homogeneous Spaces*, Princeton University Press (1963).
- [4] Ch. Benson, C. S. Gordon: *Kähler and symplectic structures on nilmanifolds*, Topology **27** (1988), no. 4, 513–518.
- [5] Ch. Benson, C. S. Gordon: *Kähler structures on compact solvmanifolds*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 971–980.
- [6] W. M. Boothby, H. C. Wang: *On contact manifolds*, Ann. of Math. (2) **68** (1958), 721–734.
- [7] S. Bosch: *Lineare Algebra*, Springer (2006).
- [8] F. Bourgeois: *Odd dimensional tori are contact manifolds*, Int. Math. Res. Notices **30** (2002), 1571–1574.
- [9] K. S. Brown: *Cohomology of Groups*, Springer (1982).
- [10] R. Campoamor-Stursberg: *Symplectic forms on six dimensional real solvable Lie algebras*, arXiv:math.DG/0507499.
- [11] R. Campoamor-Stursberg: *Some remarks concerning the invariants of rank one solvable real Lie algebras*, Algebra Colloq. **12** (2005), no. 3, 497–518.
- [12] B.-Y. Chu: *Symplectic homogeneous spaces*, Trans. Amer. Math. Soc. **197** (1974), 145–159.
- [13] L. A. Cordero, M. Fernández, A. Gray: *Symplectic manifolds with no Kähler structure*, Topology **25** (1986), no. 3, 375–380.
- [14] L. Corwin, F. P. Greenleaf: *Representations of Nilpotent Lie Groups and Their Applications*, Cambridge University Press (1990).

- [15] K. Dekimpe: *Semi-simple splittings for solvable Lie groups and polynomial structures*, Forum Math. **12** (2000), no. 1, 77–96.
- [16] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan: *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274.
- [17] A. Diatta: *Left invariant contact structures on Lie groups*, Differential Geom. Appl. **26** (2008), no. 5, 544–552.
- [18] J. Dixmier: *Sur les algèbres dérivées d’une algèbre de Lie*, Proc. Cambridge Philos. Soc. **51** (1955), 541–544.
- [19] J. Dixmier: *L’application exponentielle dans les groupes de Lie résolubles*, Bull. Soc. Math. France **85** (1957), 113–121.
- [20] S. K. Donaldson: *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. **44** (1996), no. 4, 666–705.
- [21] N. Dungey, A. F. M. ter Elst, D. W. Robinson: *Analysis on Lie Groups with Polynomial Growth*, Birkhäuser (2003).
- [22] L. Evens: *The Cohomology of Groups*, Clarendon Press (1991).
- [23] Y. Félix, St. Halperin, J.-C. Thomas: *Rational Homotopy Theory*, Springer (2001).
- [24] M. Fernández, M. J. Gotay, A. Gray: *Compact parallelizable four dimensional symplectic and complex manifolds*, Proc. Amer. Math. Soc. **103** (1988), no. 4, 375–380.
- [25] M. Fernández, A. Gray: *Compact symplectic solvmanifolds not admitting complex structures*, Geom. Dedicata **34** (1990), no. 3, 295–299.
- [26] M. Fernández, M. de León, M. Saralegui: *A six-dimensional compact symplectic solvmanifold without Kähler structures*, Osaka J. Math. **33** (1996), no. 1, 19–35.
- [27] M. Fernández, V. Muñoz: *Formality of Donaldson submanifolds*, Math. Z. **250** (2005), no. 1, 149–175.
- [28] M. Fernández, V. Muñoz: *The geography of non-formal manifolds* in O. Kowalski, E. Musso, D. Perrone: *Complex, Contact and Symmetric Manifolds*, 121–129, Prog. Math. vol. 234, Birkhäuser (2005).
- [29] M. Fernández, V. Muñoz: *An 8-dimensional non-formal simply connected symplectic manifold*, Ann. of Math. (2) **167** (2008), no. 3, 1045–1054.

- [30] S. Fukuhara, K. Sakamoto: *Classification of T^2 -bundles over T^2* , Tokyo J. Math. **6** (1983), no. 2, 311–327.
- [31] H. Geiges: *Symplectic structures on T^2 -bundles over T^2* , Duke Math. J. **67** (1992), no. 3, 539–555.
- [32] H. Geiges: *Symplectic manifolds with disconnected boundary of contact type*, Internat. Math. Res. Notices (1994), no. 1, 23–30.
- [33] H. Geiges: *Constructions of contact manifolds*, Math. Proc. Cambridge Philos. Soc. **121** (1997), no. 3, 455–464.
- [34] R. Gompf: *A new construction of symplectic manifolds*, Ann. of Math. (2) **142** (1995), no. 3, 527–595.
- [35] V. V. Gorbatsevich: *Symplectic structures and cohomologies on some solvmanifolds*, Sibirian Math. J. **44** (2003), no. 2, 260–274.
- [36] P. Griffiths, J. Harris: *Principles of Algebraic Geometry*, Wiley (1994).
- [37] St. Halperin: *Lectures on Minimal Models*, Mém. Soc. Math. France (N.S.), no. 9-10 (1983).
- [38] R. Harshavardhan: *Geometric structures of Lie type on 5-manifolds*, Ph.D. Thesis, Cambridge University (1996).
- [39] K. Hasegawa: *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc. **106** (1989), no. 1, 65–71.
- [40] K. Hasegawa: *Four-dimensional compact solvmanifolds with and without complex analytic structures*, arXiv:math.CV/0401413.
- [41] K. Hasegawa: *Complex and Kähler structures on compact solvmanifolds*, J. Symplectic Geom. **3** (2005), no. 4, 749–767.
- [42] K. Hasegawa: *A note on compact solvmanifolds with Kähler structure*, Osaka J. Math. **43** (2006), no. 1, 131–135.
- [43] A. Hatcher: *Algebraic Topology*, Cambridge University Press (2002).
- [44] A. Hattori: *Spectral sequence in the de Rham cohomology of fibre bundles*, J. Fac. Sci. Univ. Tokyo Sect. I **8** (1960), 289–331.
- [45] P. J. Hilton: *On G -spaces*, Bol. Soc. Brasil. Mat. **7** (1976), no. 1, 65–73.
- [46] P. J. Hilton: *Nilpotente Gruppen und nilpotente Räume*, Lecture Notes in Math. **1053**, Springer (1984).

- [47] R. Ibáñez, Y. Rudyak, A. Tralle, L. Ugarte: *On certain geometric and homotopy properties of closed symplectic manifolds*, *Topology Appl.* **127** (2003), no. 1–2, 33–45.
- [48] J. Kędra, Y. Rudyak, A. Tralle: *Symplectically aspherical manifolds*, *J. Fixed Point Theory Appl.* **3** (2008), no. 1, 1–21.
- [49] S. Lang: *Algebra*, 3. ed., Addison-Wesley (1993).
- [50] D. Lehmann: *Théorie homotopique des formes différentielles (d'après D. Sullivan)*, *Astérisque* **45** (1977).
- [51] R. Lutz: *Sur la géométrie des structures de contact invariantes*, *Ann. Inst. Fourier (Grenoble)*, **29** (1979), no. 1, xvii, 283–306.
- [52] J. Martinet: *Formes de contact sur les variétés de dimension 3* in C. T. C. Wall: *Proc. Liverpool Singularities Sympos. II*, 142–163, *Lecture Notes in Math.* **209**, Springer (1971).
- [53] J. Milnor: *Curvature of left invariant metrics on Lie groups*, *Advances in Math.* **21** (1976), no. 3, 293–329.
- [54] V. V. Morozov: *Classifikaziya nilpotentnykh algebrach Lie shestovo poryadka*, *Izv. Vysš. Učebn. Zaved. Matematika* **5** (1958), no. 4, 161–171.
- [55] G. D. Mostow: *Factor spaces of solvable groups*, *Ann. of Math. (2)* **60** (1954), 1–27.
- [56] G. M. Mubarakzjanov: *O rasreshimyykh algebrach Lie*, *Izv. Vysš. Učebn. Zaved. Matematika* **32** (1963), no. 1, 114–123.
- [57] G. M. Mubarakzjanov: *Classifikaziya veshchennykh struktur algebrach Lie pyatovo poryadko*, *Izv. Vysš. Učebn. Zaved. Matematika* **34** (1963), no. 3, 99–106.
- [58] G. M. Mubarakzjanov: *Classifikaziya rasreshimyykh struktur algebrach Lie shestovo poryadka a odnim nenilpoentnym bazisnym elementom*, *Izv. Vysš. Učebn. Zaved. Matematika* **35** (1963), no. 4, 104–116.
- [59] I. Nakamura: *Complex parallelisable manifolds and their small deformations*, *J. Differential Geometry* **10** (1975), 85–112.
- [60] A. Newlander, L. Nirenberg: *Complex analytic coordinates in almost complex manifolds*, *Ann. of Math. (2)* **65** (1957), 391–404.
- [61] K. Nomizu: *On the cohomology of homogeneous spaces of nilpotent Lie Groups*, *Ann. of Math. (2)* **59** (1954), 531–538.

- [62] T. Nôno: *On the singularity of general linear groups*, J. Sci. Hiroshima Univ. Ser. A **20** (1956/1957), 115–123.
- [63] J. Oprea, A. Tralle: *Symplectic Manifolds with no Kähler Structure*, Lecture Notes in Math. **1661**, Springer (1997).
- [64] A. L. Onishchik, E. B. Vinberg: *Lie Groups and Lie Algebras III*, Springer (1994).
- [65] D. Quillen: *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295.
- [66] M. S. Raghunathan: *Discrete Subgroups of Lie Groups*, Springer (1972).
- [67] M. Saitô: *Sur certaines groupes de Lie résolubles – parties I et II*, Sci. Papers Coll. Gen. Ed. Univ. Tokyo **7** (1957), 1–11, 157–168.
- [68] S. M. Salamon: *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), no. 2-3, 311–333.
- [69] H. Sawai, T. Yamada: *Lattices on Benson-Gordon type solvable Lie groups*, Topology Appl. **149** (2005), no. 1-3, 85–95.
- [70] N. Steenrod: *The Topology of Fibre Bundles*, Princeton University Press (1951).
- [71] D. Sullivan: *Geometric Topology – Part I*, M.I.T. Press (1970).
- [72] D. Sullivan: *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–331.
- [73] W. P. Thurston: *Some simple examples of compact symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976), no. 2, 467–468.
- [74] A. Tralle: *On solvable Lie groups without lattices* in M. Fernández, J. A. Wolf: *Global Differential Geometry*, 437–441, Contemp. Math. **288**, Amer. Math. Soc. (2001).
- [75] P. Turkowski: *Solvable Lie algebras of dimension six*, J. Math. Phys. **31** (1990), no. 6, 1344–1350.
- [76] V. S. Varadarajan: *Lie Groups, Lie Algebras, and Their Representations*, Springer (1984).
- [77] F. W. Warner: *Foundations of Differentiable Manifolds and Lie Groups*, Glenview (1971).
- [78] Ch. A. Weibel: *An Introduction to Homological Algebra*, Cambridge University Press (1997).

- [79] T. Yamada: *A pseudo-Kähler structure on a nontoral compact complex parallelizable solvmanifold*, *Geom. Dedicata* **112** (2005), 115-122.

Index

- Almost Abelian Lie Group, 32
- Almost Nilpotent Lie Group, 32
- Ascending Series, 21

- Characteristic Polynomial, 119
- Chevalley-Eilenberg Complex, 24
- Cohomologically Symplectic Lie Algebra, 68
- Completely Solvable
 - Lie Algebra, 28
 - Lie Group, 28
- Contact Manifold, 17

- Derived Series, 21
- Descending Series, 21
- Differential Graded Algebra (DGA), 1
- Donaldson Submanifold, 12

- Exponential Lie Group, 28

- Formality of a
 - Differential Graded Algebra, 4
 - Manifold, 4

- Hard Lefschetz Manifold, 10

- Jordan Decomposition, 36

- Kähler Manifold, 10

- Lattice, 22, 25
- Lattice Subgroup, 23
- Lefschetz Manifold, 10

- Massey Product, 5
- Minimal Model of a
 - Differential Graded Algebra, 2
 - Manifold, 3

- Simplicial Complex, 97
- Minimal Polynomial, 119
- Mostow Bundle, 26

- Nijenhuis Tensor, 10
- Nilmanifold, 22
- Nilpotent
 - Group, 21
 - Lie Algebra, 21
 - Module, 98
 - Space, 98
- Nilshadow, 35

- PL Form, 97

- Quasi-Isomorphism, 2

- Root, 119

- Semidirect Product
 - of (Lie) Groups, 31
 - of Lie Algebras, 31
- Semisimple Splitting, 35
- Solvable
 - Group, 21
 - Lie Algebra, 21
- Solvmanifold, 25
- Structure Constants, 22
- Symplectic Manifold, 9

- Unimodular Lie Group, 21

Ich versichere, daß ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken dem Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; daß diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; daß sie – abgesehen von den unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist sowie, daß ich eine solche Veröffentlichung vor Abschluß des Promotionsverfahrens nicht vornehmen werde.

Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Herrn Prof. H. Geiges betreut worden.

Teilpublikationen:

- (1) Die in Kapitel 2 beschriebenen Ergebnisse sind von mir bereits auf dem Preprint-Server *arXiv.org e-print archives* (<http://xxx.lanl.gov/>) veröffentlicht worden.